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STRUCTURE CONTROL IN QUEUING AND RELIABILITY MODELS

The optimal strategy of the structure control in queue and reliability model is studied by using controlled semi-Markov processes. The optimal strategy has been proved to be looked for in the class of threshold strategies.

Keywords: *controlled semi-Markov process, degenerated and threshold control strategies, queuing systems, reliability models and service.*

Introduction

Queue models are quite frequently used for adequate description of the functioning processes of various real technical and economic systems. The specific features of queuing systems (QS) are the presence of an input stream of requests, serving devices, a queue which distributes the requests to serving devices.

The tasks of QS analysis are similar to the service tasks, which are an integral part of the mathematical theory of reliability, as the abovelisted components reside in the process of functioning of any technical system. The stream of service requests is composed by the elements of a system (subsystem) failed during operation and requiring recovery and service channels are the service teams responsible for repair.

When analyzing QS and reliability, the optimization problems hold an important place. Considering the characteristic features of systems under study, an optimization task can be set for all the components which determine the system. In particular, the control of system's structure is considered as a change of the number of involved service channels (service teams) and of the number of waiting facilities.

Following the general principles of the control task assignment, let us define that *a control object* is a controlled process describing the system evolution with time, *control strategies* is a set of decisions and the decision rule, and *a measure*, specifying the control quality.

The task is to define the control strategy, for which a measure specifying the control quality takes on an extremum value.

This paper represents the model of the controlled semi-Markov process [1] necessary to construct an optimal strategy for the system structure control. The controlled semi-Markov process is defined as the process with two consecutive components $X(t)=\{\xi(t),u(t)\}$, $\xi(t)\in E$, $u(t)\in U$, where E is a state space, U is a control space, where the moments of discontinuity of the components do coincide, and at these moments of the state change the process possesses a Markov property.

Initial probabilistic characteristics:

• Semi-Markov kernel $Q_{ij}(t,u) = P\{\xi_{n+1}=j, \theta_{n+1}<t/\xi_n=i, u_{n+1}=u\}$, which is equal to a conditional probability of the fact that the following value of the first component is j, $\xi_{n+1}=j$, and this transition shall happen up to the moment t, $\theta_{n+1}<t$, provided that the previous value of

the first component is equal to i, $\xi_n=i$, and then the decision is taken u, $u_{n+1}=u$;

• The measures $G_i(B)$, $i\in E$, $u\in U$, $B\in A$, which define the control strategy (decision rule). These measures are defined on measurable control spaces (U, A)

Quality indicator is defined by the functions $R_{ij}(t,u)$, $i,j\in E$, $u\in U$, $0\le t<\infty$, which are equal to the mathematical expectation of the accumulated effect for the duration of QS being in state i provided that in time t it shall pass into state j and then the decision u is taken.

The above integrated characteristics let us define the mathematical expectation of the accumulated effect for the duration of the process being in state i

$$s_i=\sum_{j\in E}\int_{u\in U_i}\int_0^\infty R_{ij}(x,u)dQ_{ij}(x,u)G_i(du) \qquad (1)$$

and the mathematical expectation of the process continuously being in state i

$$m_i=\sum_{j\in E}\int_{u\in U_i} xdQ_{ij}(x,u)G_i(du). \qquad (2)$$

Papers [1, 2, 3] prove the following statements:

• If an imbedded Markov chain is irreducible, then for the mathematical expectation $S_i(t)$ accumulated during the time t of the effect, provided it starts from the state $i\in E$, an asymptotic equation holds true with $t\to\infty$ $S_i(t)=St+o(t)$;

• Dependence on the initial characteristics of the functional S is defined by the equation

$$S\left(\vec{G}\right)=\frac{\sum_{i=1}^{N}s_i\pi_i}{\sum_{i=1}^{N}m_i\pi_i}, \qquad (3)$$

where π_i, $i\in E$ denotes a stationary distribution of the imbedded Markov chain, which is a normalized solution of the algebraic system of equations [4]

$$\pi_i=\sum_{j=E}\pi_j p_{ji},\quad \sum_{j=E}\pi_j=1,$$

$$p_{ij}=\int_{U_i}\left(\lim_{t\to\infty}Q_{ij}(t,u)\right)G_i(du); \qquad (4)$$

• The functional $S\left(\vec{G}\right)$ is a linear fractional functional with regard to the distributions $\vec{G}=\left(G_1,G_2,...,G_N\right)$, specifying the Markov control strategy;

• If the extremum of a linear fractional functional does exist on the set of the acceptable strategies and all the degenerated strategies are acceptable, then this extremum shall be reached on the set of degenerated strategies.

These mathematical results shall be used for analysis of the exact controlled queue models and models of reliability. Particularly, the further calculations shall be made directly for degenerated control strategies.

Now then, the definition of a Semi-Markov kernel and formulas (1) – (4) prescribe the sequence of the stages for the analysis of the exact models.

Task definition

This section will be dedicated to the analysis of QS which receives a recurrent flow of demands or a recovery process at its input. Intervals between the neighboring moments of the demands entries shall be specified through ξ_i, $i=1,2,\ldots k,\ldots$, and the function of distribution of these intervals shall be specified through $F(x)=P\{\xi_i<x\}$, $F(0+0)=0$. The latter condition means the ordinariness of the arrival stream. The durations of service η have exponential distribution $G(x)=P\{\eta<x\}=1-e^{-\mu x}$, $x\ge0$. There is no queue in the system. Unlike classical arrangements, let us consider the system structure to be variable – the number of functioning channels varies in the system, but it can not exceed the value η, $0<\eta<\infty$.

The decision to change the number of functioning channels in the system is taken in the moments t_k – the moments of entry of a recurrent demand. It means that within the time interval $[t_k, t_{k+1})$ there are no demands to the system, but at the moment of t_{k+1} there is only *one* demand entry. The number of waiting facilities is equal to zero, that is why the demand received at the moment of t_{k+1} is lost, if at the moment of its entry all the available service channels are occupied, and it is taken into service if there is an available free channel at the moment of t_{k+1}. The availability of a free service channel at the moment of t_{k+1} is linked to the release of the occupied channels within the time interval $[t_k, t_{k+1})$ and to the decisions taken about the number of functioning channels at the moment of t_k.

Let us assume that if at the moment of t_k a free channel is connected, all the channels released within the period $[t_k, t_{k+1})$ are disconnected; if at the moment of t_k a free channel is not connected, then among the channels released within the period $[t_k, t_{k+1})$ the first channel remains active, and all the other channels disconnect; if within the period $[t_k, t_{k+1})$ none of the channels release, any requirement received at the moment of t_{k+1} is lost.

Let us use $u_0=0$ to specify the decision to retain the number of channels, equal to the number of demands present in the system at the moment of t_k+0, using $u_1=1$ decision to retain one more additional free channel along with the channel occupied with service operation of the demands current at the moment of t_k+0 (it is sufficient to retain just one free channel as at the moment of t_{k+1} there will be only one demand entry).

If at the moment of decision making there is i of demands in QS, then with the probability $0\le p_i\le1$ the decision $u_0=0$ is made, and with the probability $0\le q_i\le1$ the decision $u_1=1$, $p_i+q_i=1$, $i=0,1,2,\ldots,n$ is made. With $i=n$ we have $p_n=1$, $q_n=0$ due to the limited total number of the channels of value n. Using mathematical terminology, at first, a class of randomized control strategies is used [1, 2] under the task definition.

Let us enter the cost performance characteristics which define the functional, specifying the quality of performance and control. Let us assume that c_0 is the profit per one served demand; c_1 is a pay for one hour of the involved channel operation; c_2 is a pay for one hour of a free channel downtime, c_3 is a pay for the loss of one demand.

The model described above fully stays within a model of discrete control of a semi-Markov process [1].

Task solution

1. *Construction of a control object*. The control object shall be a semi-Markov process $X(t)$, describing the evolution of the respective queuing system with time. In order to define it, let us introduce the sequence $t_0=0$, t_k, $k=1,2,\ldots$, $t_k \le t_{k+1}$ of neighboring moments of the demands entries into the system. Let us define a stochastic process $X(t)$, with $X(t)=i$, $t\in[t_k,t_{k+1})$, if at the moment of $t_k + 0$ there are i requests under operation. This means that above introduced stochastic process takes the values from the range $E=\{0,1,2,\ldots,n\}$. We shall note that the state $i=0$ is realized when there were no demands, and when the arrived demand was not lost. If we know the number of demands in the system at the moment of entry of a recurrent demand, then under the taken expectations this process is a Markov process, as the distribution of the time interval followed by the next request, does not depend on the past, and the distribution of the number of the demands served in this time interval does not depend on the past either, due to the absence of consequence of the exponential distribution. Therefore, this is true that the process $X(t)$ is semi-Markov.

2. *Definition of a semi-Markov matrix.* For the model under analysis, the elements of a semi-Markov matrix are defined by the equations with $i\in E$

$$Q_{01}(t,1)=F(t),$$

$$Q_{ij}(t,1)=\int_0^t C_i^{j-1}e^{-(j-1)\mu x}(1-e^{-\mu x})^{i-j+1}dF(x),$$

$$i=(0,1,\ldots,n\text{-}1),\ \ 0<j\le i+1;$$

$$Q_{ii}(t,0)=\int_0^t [e^{-i\mu x}+ie^{-(i-1)\mu x}(1-e^{-\mu x})]dF(x),$$

$$i=(0,1,\ldots,n);$$

$$Q_{ij}(t,0)=\int_0^t C_i^{j-1}e^{-(j-1)\mu x}(1-e^{-\mu x})^{i-j+1}dF(x),$$

$$i=(2,\ldots,n),\ \ 1\le j\le i\text{-}1. \qquad (5)$$

Other elements of a semi-Markov matrix are equal to zero.

Let us clarify the equations (5). If we know the taken decision, then we also know the number of requests being served at the moment of decision-making – the moment of the request entry. If we know that the next demand shall arrive at time x, then, due to a wonderful property of the absence of consequence of exponential distribution, at the moment of a recurrent request the number of unserved demands has the Bernoulli distribution with a parameter $e^{-\lambda x}$. It leads to formulas (5).

3. *Description of control space and strategies space*. For each state $i\in E=\{0,1,2,\ldots,n\}$ the control spaces $U_i=\{0,1\}$, $i\ne n$, $U_n=\{0\}$, are composed of two possible decisions: to link a free channel, or not to link it (except for the state $i=n$, when there are no free functioning channels available), therefore, randomized strategies are defined by probabilistic distributions (p_i,q_i), $p_i\ge 0$, $q_i\ge 0$, $p_i+q_i=1$, $i\ne n$, $q_n=0$. Thus, any of the randomized strategies can be expressed by a vector $(p_0, p_1, \ldots, p_{n-1}, p_n = 1)$. Space of degenerated strategies, with consideration of $p_n = 1$ can be equated with a set of n-dimensional vectors of zeros and ones, containing 2^n of elements. We shall further define any degenerated strategy by $(n+1)$-dimensional dyadic vector, in which the one in the k-th bit means that in the state k with probability one, the decision is taken to link a free service channel, and a zero in the k-th position means that in the state k with probability one, the decision is taken not to link a free service channel. Let us note that this dyadic vector does always have the value of the last bit equal to zero.

4. *Construction of the functional*. Let us use $A_{ij}(t,u)$, $i,j\in E$, $t\ge 0$, $u=0$, 1 to specify the event of the process in the state i, passed during the time t into the state j and the decision u was taken.

If in the state $i\in E$ the decision $u=0$ is taken and the process passed into the state j, then:

• with $i=0$ there are no demands in service at the beginning of the period, none of the channels is activated, the arrived demand is lost. Consequently, under the listed conditions, a mathematical expectation of the accumulated effect, expressed through income and efforts, is equal to

$$R_{00}(t,0)=c_3. \qquad (6);$$

• with $i=1,2,\ldots,n$, $j=i$ at the beginning of the period the i of demands are in service and there are no free channels. The transition into the state $j=i$ is possible in two cases: either during this period no demand is completely served (in this case the arrived demand is lost) – the event B_0, or during this period one demand is completely served and the arrived demand entered the service – the event B_1. Thus under the accepted specifications the equations (5) lead to

$$p_0=P\{B_0/A_{ii}(t,0)\}=\frac{e^{-i\mu t}}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})},$$
$$p_1=P\{B_1/A_{ii}(t,0)\}=\frac{ie^{-(i-1)\mu t}(1-e^{-\mu t})}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})}. \qquad (7)$$

Further onб let us use v to specify the number of demands after service, and ζ could be used to specify the total time of channels operation on the period between the neighboring Markov moments of the demand entries. Then under the accepted conditions, the mathematical expectation of the accumulated effect, expressed through income and efforts, is equal to

$$R_{ii}(t,0)=c_0M[\nu/A_{ii}(t,0)]+C_{ii}(t,0)+c_3(1-M[\nu/A_{ii}(t,0)]), \qquad (8)$$

where $C_{i,i}(t,0)$ is the mathematical expectation of the efforts on the operation and the down time of a service channel on the period under consideration under the same conditions. That is why we have

$$M[\nu / A_{ii}(t,0)] = \frac{ie^{-(i-1)\mu t}(1-e^{-\mu t})}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})},$$
$$1-M[\nu / A_{ii}(t,0)] = \frac{e^{-i\mu t}}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})}. \quad (9)$$

Considering that these efforts are proportional to the total time of operation of the service channels, we can write

$$C_{ii}(t,0) = c_1 M(\zeta / A_{ii}(t,0)) + c_2[t - M(\zeta / A_{ii}(t,0))],$$

where $M(\zeta/A_{ii}(t,0))$ is the mathematical expectation of the total time of the operation of the service channels on the considered period provided the transition of the process from the state $i\in E$ to the state $j\in E$ during t time. Let us further note that under the accepted statements, the process defined as the number of requests in the system is described by the Markov process of death with the transitions rates $\mu_k=k\mu$, $j-1\le k\le i$. That is why the total time of operation of service channels on the considered period providing the process transition from the state $i\in E$ to the state $j\in E$ during t time is the integral of the path $\xi(t,\omega)$ of the Markov process of death with the transitions rates $\mu_k=k\mu$, $j-1\le k\le i$, for which either $\{\xi(0)=i, \xi(t)=i\}$ condition, or $\{\xi(0)=i, \xi(t)=i-1\}$ condition are fulfilled.

The paper [2] contains the correlations for integral conditional mathematical expectations, and these correlations lead to the formulas

$$M(\int_0^t \xi(x,\omega)dx / \xi(t)=k,\ \xi(0)=n) =$$
$$= kt + (n-k)\frac{\frac{1}{\mu}\left(1-e^{-\mu t}\right) - te^{-\mu t}}{1-e^{-\mu t}},\quad k \ge n. \quad (10)$$

Then, with account of correlations (7) and (10) we can write

$$\begin{aligned} C_{ii}(t,0) &= c_1 M(\zeta / A_{ii}(t,0)) + c_2[it - M(\zeta / A_{ii}(t,0))] = \\ &= c_1[p_0 M(\int_0^t \xi(x,\omega)dx / \xi(t)=i,\ \xi(0)=i)) + \\ &+ p_1 M(\int_0^t \xi(x,\omega)dx / \xi(t)=i\text{-}1,\ \xi(0)=i)] + \\ &+ c_2[it - p_0 M(\int_0^t \xi(x,\omega)dx / \xi(t)=i,\ \xi(0)=i)) - \\ &- p_1 M(\int_0^t \xi(x,\omega)dx / \xi(t)=i\text{-}1,\ \xi(0)=i)] = \\ &= c_1\frac{itC_i^{i-i}e^{-i\mu t} + C_i^{i-(i-1)}e^{-(i-1)\mu t}(1-e^{-\mu t})\left((i-1)t+\left(\frac{1}{\mu}-\frac{te^{-\mu t}}{1-e^{-\mu t}}\right)\right)}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})} + \\ &+ c_2\frac{C_i^{i-(i-1)}e^{-(i-1)\mu t}\left(t-\frac{1}{\mu}(1-e^{-\mu t})\right)}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})}. \end{aligned} \quad (11)$$

By combination of the equations (8), (9) and (11), we can get

$$\begin{aligned} R_{ii}(t,0) &= c_3 + (c_0 - c_3)\frac{ie^{-(i-1)\mu t}(1-e^{-\mu t})}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})} + \\ &+ c_2\frac{ie^{-(i-1)\mu t}\left(t-\frac{1}{\mu}(1-e^{-\mu t})\right)}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})} + \\ &+ c_1[\frac{ite^{-i\mu t}}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})} + \\ &+ \frac{ie^{-(i-1)\mu t}(1-e^{-\mu t})\left((i-1)t+\left(\frac{1}{\mu}-\frac{te^{-\mu t}}{1-e^{-\mu t}}\right)\right)}{e^{-i\mu t}+ie^{-(i-1)\mu t}(1-e^{-\mu t})}]; \end{aligned} \quad (12)$$

• with $2\le i<n$, $0<j\le i-1$ in the period between the neighboring moments of demand entries into the system, only the demands are being served and thus the number of the demands reduces. The number of the served demands is positive and is equal to $i-j+1\ge 2$. Consequently, at the moment of a new demand entry, there will be a free channel available and the demand won't be lost. Thus,

$$R_{ij}(t,0) = c_0(i-j+1) + C_{ij}(t,0), \quad (13)$$

where using $C_{i,j}(t,0)$ we shall specify the mathematical expectation of the efforts for operation and down time of service channels on the considered period provided there occurred the event $A_{ij}(t,0)$. The number of the served demands is equal to $i-j+1\ge 2$, that is why the mathematical expectation of the total time of service operation, defined by the formula (10), is equal to

$$M\left(\int_0^t \xi(x,\omega)dx / \xi(t)=j,\ \xi(0)=i\right) =$$
$$= (j-1)t + (i-j+1)\frac{\frac{1}{\mu}(1-e^{-\mu t}) - te^{-\mu t}}{1-e^{-\mu t}}, \quad (14)$$

as a newly arrived demand just starts being served.

At the same time, the first released channel remained being linked but was in a down state up the moment of a new demand entry. The paper [2] defines the mathematical expectation $M(\xi_{i,t} / \xi(t) = j-1, \xi(0)=i)$ of the first released channel operation time

$$M(\xi_{i,t}/\xi(t)=j-1,\ \xi(0)=i) = \int_0^t \left(\frac{e^{-\mu x}-e^{-\mu t}}{1-e^{-\mu t}}\right)^{i-j+1} dx. \quad (15)$$

Using the equations (14) and (15), we get with $2\le i<n$, $0<j\le i-1$

$$C_{ij}(t,0)=c_1M\left(\int_0^t \xi(x,\omega)dx\,/\,\xi(t)=j,\xi(0)=i\right)+$$
$$=c_2\left[t-M(\xi_{i,t}\,/\,\xi(t)=j-1,\xi(0)=i)\right]=$$
$$=c_1\left((j-1)t+(i-j+1)\frac{\frac{1}{\mu}(1-e^{-\mu t})-te^{-\mu t}}{1-e^{-\mu t}}\right)+$$
$$+c_2\left(t-\int_0^t\left(\frac{e^{-\mu x}-e^{-\mu t}}{1-e^{-\mu t}}\right)^{i-j+1}dx\right). \qquad (16)$$

Then with $2\le i<n$, $0<j\le i-1$ from (13) и (16) we shall have

$$R_{ij}(t,0)=c_0(i-j+1)+$$
$$+c_1\left((j-1)t+(i-j+1)\frac{\frac{1}{\mu}(1-e^{-\mu t})-te^{-\mu t}}{1-e^{-\mu t}}\right)+$$
$$+c_2\left(t-\int_0^t\left(\frac{e^{-\mu x}-e^{-\mu t}}{1-e^{-\mu t}}\right)^{i-j+1}dx\right). \qquad (17)$$

So we have calculated all desired mathematical expectations under the decision u=0. It is not necessary to perform these calculations for other parameter values $i,j\in E$, because the respective elements of a semi-Markov kernel (5) are equal to zero.

If in the state $i\in E$ the decision u=1 is taken and the process passes to the state j, then:

• with i=0 at the beginning of the period there are no demands in service, none of the channels is activated, one free channel is linked, the arrived demand is not lost, as there was a free channel available. Consequently, under the listed conditions, a mathematical expectation of the accumulated effect, expressed through income and efforts is equal to

$$R_{01}(t,1)=c_2t; \qquad (18)$$

• with $i=n$ a decision on the fitting of a free channel can not be made. With $n-1\ge i\ge 1$ at the beginning of the period the demands in the amount of i are being served and there is one free channel, during the period the demands in the amount of $i-j+1\ge 0$ are completely served and, at last, the demand arrived in the end of the period is not lost. Thus, under the listed conditions, the mathematical expectation of the accumulated effect expressed through income and efforts is equal to $R_{ij}(t,1)=c_0(i-j+1)+C_{i,j}(t,1)+c_2t$, where, as before, the $C_{i,j}(t,1)$ is used to specify mathematical expectation of the efforts on the service channels operation on the considered period under the same conditions, i.e. the event $A_{ij}(t,1)$occurred.

Using the equation (10), with $n\ge i+1\ge j\ge 1$ we get the mathematical expectation of the total time of service channel operation

$$C_{i,j}(t,1)=c_1\left((j-1)t+(i-j+1)\frac{\frac{1}{\mu}(1-e^{-\mu t})-te^{-\mu t}}{1-e^{-\mu t}}\right).$$

It leads to

$$R_{ij}(t,1)=c_0(i-j+1)+c_2t+$$
$$+c_1\left((j-1)t+(i-j+1)\frac{\frac{1}{\mu}(1-e^{-\mu t})-te^{-\mu t}}{1-e^{-\mu t}}\right). \qquad (19)$$

It is not necessary to calculate the rest of mathematical expectations, as the respective elements of a semi-Markov kernel are equal to zero.

5. *Calculation of mathematical expectations of time of the process continuously being at a fixed state.* For the model under analysis, mathematical expectations $m_i(u)$, $i\in E$, u=0,1 with consideration of (2) and (5) are defined by the equations

$$m_i(u)=\int_0^\infty tdF(t)=M,\ i\in E,\ u=0,1. \qquad (20)$$

6. *Calculation of mathematical expectations of income.* Mathematical expectations $s_i(u)$, $i\in E$, u=0,1 of the income for the time of the process continuously being with consideration of (1), (5), (12) and (17)-(19) are defined by the equations

$$s_i(1)=c_0i\int_0^\infty(1-e^{-\mu t})dF(t)+c_2\int_0^\infty tdF(t)+$$
$$+c_1i\int_0^\infty te^{-\mu t}dF(t)+c_1\frac{i}{\mu}\int_0^\infty\left(1-(1+\mu t)e^{-\mu t}\right)dF(t)=$$
$$=c_0i\int_0^\infty(1-e^{-\mu t})dF(t)+c_2\int_0^\infty tdF(t)+c_1\frac{i}{\mu}\int_0^\infty\left(1-e^{-\mu t}\right)dF(t)=$$
$$=\int_0^\infty i\left(c_0+\frac{c_1}{\mu}\right)\left(1-e^{-\mu t}\right)dF(t)+c_2\int_0^\infty tdF(t), \qquad (21)$$

$$s_i(0)=ic_0\int_0^\infty\left(1-e^{-\mu t}\right)dF(t)+c_3\int_0^\infty e^{-i\mu t}dF(t)+$$
$$+\frac{ic_1}{\mu}\int_0^\infty\left(1-e^{-\mu t}\right)dF(t)+c_2\int_0^\infty\left(t-\frac{\left(1-e^{-i\mu t}\right)}{i\mu}\right)dF(t),\ 1\le i\le n,$$
$$s_0(0)=c_3,\ \ i=0. \qquad (22)$$

7. *Calculation of steady-state probabilities of an imbedded Markov chain.*

With consideration of correlations (5) for the matrix elements P of transition probabilities the following equations follow

$$p_{01}(1) = Q_{01}(\infty,1) = 1,$$

$$p_{ij}(1) = Q_{ij}(\infty,1) = \int_0^\infty C_i^{j-1} e^{-(j-1)\mu x}(1 - e^{-\mu x})^{i-j+1} dF(x),$$
$$i = (0,1,...,n-1),\ 0 < j \le i+1;$$

$$p_{ii}(0) = Q_{ii}(\infty,0) = \int_0^\infty [e^{-i\mu x} + ie^{-(i-1)\mu x}(1-e^{-\mu x})]dF(x),$$
$$i = (0,1,...,n);$$
$$p_{ij}(0) = Q_{ij}(\infty,0) = \int_0^\infty C_i^{j-1} e^{-(j-1)\mu x}(1 - e^{-\mu x})^{i-j+1} dF(x),$$
$$i = (2,...,n),\ 1 \le j \le i-1. \qquad (23)$$

Let us further consider the set of degenerated strategies, for which the decision to link a free channel with probability one is taken for the states i, $0 \le i \le k < n$. These degenerated strategies are expressed by dyadic vectors, first k of components of which take on the value of one, the k+1-th component takes on the value of zero, and the rest components takes on any values except u_n=0, i.e. all the sets of degenerated strategies under different $0 \le k \le n$ divide into disjoint sub-sets of the type (1,1,…, 1, 0, u_{k+1},…, u_{n-1}, 0). For any strategy of this set u_k=0 and $p_{k,k+1}(0)$=0.

Thus, under $0 \le k \le n$, stationary probabilities of the states of the imbedded Markov chain π_i, $i \in E$ correspond to the system of algebraic expressions (5), where the transition probabilities are defined by the equations (23).

Under $p_{k,k+1}(0)$=0 transitions from the states E_0={0, 1, … k} into the states from the set E_1={k+1, …, n} are impossible. Therefore, the set E_1={k+1, …, n} is the set of nonexistent states, and the set $E_0 = \{0,1,...k\}$ forms a closed class of communicating states [4] and there is the only steady-state distribution, for which the following correlations are valid:

$$\pi_i > 0,\ i \in E_0 = (0,1,...,k),\ \pi_i = 0,$$
$$i \in E_1 = (k+1,...,n),\ \sum_{i=0}^{k} \pi_i = 1.$$

To define steady-state distribution of π_i>0, $i \in E_0$, it is necessary to find a system's normalized solution

$$\pi_0 = \sum_{j=0}^{k-1} \pi_j p_{j0}(1) + \pi_k p_{k0}(0),$$
$$\pi_i = \sum_{j=i-1}^{k-1} \pi_j p_{ji}(1) + \pi_k p_{ki}(0),\ i = 1,2,...,k, \qquad (24)$$

which shall be specified by $\pi_i^{(k)},\ 0 \le k \le i \le n.$

The equation (24) proves that for any degenerated strategy of the selected sets, the steady-state distributions of the impeded Markov chain are common.

8. *Calculation of the quality indicator and selection of optimal control strategy.* For the case under consideration, the set of process states forms a single class of communicating states. That is why we shall use the equation (3) for calculation of quality indicator. Let us plug a normalized solution of algebraic system of equations (24) and expressions (20) – (22) into (3), then we have $S^{(0)}=c_3$, k=0 and under $1 \le k \le n$

$$S^{(k)} = \frac{\sum_{i=0}^{k-1} \pi_i^{(k)} s_i(1) + \pi_k^{(k)} s_k(0)}{\int_0^\infty x dF(x)} =$$
$$= \frac{\sum_{i=0}^{k-1} \pi_i^{(k)} \left(\int_0^\infty i\left(c_0 + \frac{c_1}{\mu}\right)\left(1 - e^{-\mu t}\right) dF(t) + c_2 \int_0^\infty t dF(t) \right)}{\int_0^\infty x dF(x)} +$$
$$+ \frac{\pi_k^{(k)}}{\int_0^\infty x dF(x)} (kc_0 \int_0^\infty \left(1 - e^{-\mu t}\right) dF(t) + c_3 \int_0^\infty e^{-i\mu t} dF(t) +$$
$$+ \frac{kc_1}{\mu} \int_0^\infty \left(1 - e^{-\mu t}\right) dF(t) + c_2 \int_0^\infty \left(t - \frac{\left(1 - e^{-k\mu t}\right)}{k\mu} \right) dF(t)).$$

Let us choose the max($S^{(0)}$, $S^{(10)}$, …, $S^{(n-1)}$, $S^{(n)}$)=$S^{(k0)}$ and the number k_0, this maximum is achieved at. Thus, there is a threshold optimal strategy, for which it is necessary to link a free service channel in the states i=0,1,…,k_0–1, and in the state i=k_0 it is not necessary to link a free service channel.

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