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MATHEMATICAL MODEL OF DEPENDABILITY FOR A COMPLEX "FACILITY OF PROTECTION – SAFETY SYSTEM" IN CASE OF FUZZY INITIAL INFORMATION

The given paper considers the mathematical model of dependability for a complex "facility of protection – safety system" with a periodically controlled safety system. A random fuzzy process of recovery is used for construction of a dependability mathematical model, which takes into account the uncertainty of model parameters. A fuzzy estimation has been obtained for the mathematical expectation of a complex's failure-free operating life, and the calculation of mean time to failure of a complex "facility of protection – safety system" has been exemplified.

Keywords: *dependability, random fuzzy variables, Zadeh's extension principle, membership function, fuzzy mean time, estimation of dependability performance, defuzzification.*

Introduction

The complexes consisting of a protected facility and a safety system are applied where it is necessary to provide safe operation of potentially hazardous facilities. As an example, it is possible to consider the nuclear industry. A safety system is intended to shift emergencies into non-emergencies in case of violation of normal operation of a protected facility, i.e. to parry a protected facility failure. The considered system possesses the following typical features: it has a vertical hierarchy – the facility is under the control of a safety system; the safety system possesses the right of intervention to prevent potentially hazardous changes in a facility of protection; there is an interdependence of actions – the success of system actions as a whole depends on the behavior of all system elements. Whence it follows that a facility of protection and a safety system should be considered as a whole, as a common automated technological complex "facility of protection – safety system" (ATC FP-SS). Works [1, 2] are devoted to the development of a dependability mathematical model of such complex.

Uncertainty of analysis results caused by various reasons can take place when analyzing system dependability. In this work, we shall consider the uncertainty of results of dependability analysis caused by uncertainty of initial data. The given type of uncertainty arises as parameters of a mathematical model used for the analysis of system dependability cannot be precisely known owing to insufficiency of data and variability of characteristics.

Several approaches differing from each other have been developed for modeling various aspects of uncertainty, such as a theoretical probabilistic approach, fuzzy sets and measures, and some other methods. Discussion of their distinctions and advantages can be found in study [3]. Here we shall use a combination of a theoretical probabilistic approach and fuzzy measures for construction of a dependability mathematical model considering uncertainty of model parameters. The similar approach is stated in a number of studies [4 – 8]. In the present study following Lu [4, 5], we shall use random fuzzy variables as they allow to create in a most simple fashion the mathematical model of dependability of ATC FP-SS considering uncertainty of initial data. At the analysis of a dependability mathematical model, we switch over from random variables to random fuzzy variables, and, hence, there is a necessity for consideration of random fuzzy process of recovery, which is examined in works [9, 10].

Formalization and task solution

In development of dependability mathematical models, an approach is generally accepted when operating times and recovery times are described by means of random variables. For example, it is possible to consider time to failure χ with the function of distribution $F_\chi(t;\vec{\lambda})$, where $\vec{\lambda}$ is a vector of distribution parameters. However, as a rule, exact values of parameters $\vec{\lambda}$ by virtue of those or other reasons are not known, and there is uncertainty of model parameters that in turn leads to uncertainty of values of a required dependability parameter. For the quantitative account of this uncertainty, we shall take advantage of the mathematical tool of random fuzzy variables [4,5]. The essence of the used approach consists in the fact that the measure of likelihood is attributed to random variables. We shall define random fuzzy times to failure and recovery times using the approach considered in [4]. To set a random fuzzy variable χ, we shall specify the family of probabilistic distributions $\left\{F_\chi\left(t;\vec{\lambda}(\theta)\right),\theta\in\Theta\right\}$ on a probabilistic space (Ω, A, P), where $\vec{\lambda}$ is a fuzzy vector defined on the space of likelihood $\left(\Theta,\Pi,\mathrm{Cr}\right)$ to which the membership function $\mu_{\vec{\lambda}}(\vec{x})$ corresponds.

We shall use random fuzzy variables instead of random variables for construction of the dependability mathematical model of a complex "facility of protection – safety system". The essence of the given approach consists in the fact that we shall attribute a measure of likelihood to random variables.

Let us consider that values χ_i are random fuzzy quantities, each of which is specified on a corresponding space of likelihood $\left(\Theta_i,\mathrm{P_i},Cr_i\right)$, and all of them are independent, positive and equally distributed. Similarly, γ_i is random fuzzy quantities, each of which is specified on a corresponding space of likelihood $\left(\Theta_i',\mathrm{P'_I},Cr_i'\right)$, and they are also independent, equally distributed and positive. The number of a process cycle of protected facility operation when there was a failure (an accident) we shall designate as ν. Besides, we consider that ν is a positive random fuzzy integer (i.e. a random fuzzy quantity which possesses only positive integers), defined on a space of likelihood $\left(\Theta'',\mathrm{P''}Cr''\right)$. And, finally, let us suppose that sequences $\{\chi_i, i\geq1\}$ and $\{\gamma_i, i\geq1\}$ are mutually independent, and ν is independent on sequences $\{\chi_i, i\geq1\}$ and $\{\gamma_i, i\geq1\}$.

The mean time to the first failure of a complex can be written as follows [2]:

$$\omega=\sum_{i=1}^{\nu-1}(\chi_i+\gamma_i)+\chi_\nu,$$

where χ_ν is the time to failure on that cycle of regeneration on which a protected facility failed. Thus, ω is the random fuzzy quantity defined on a space of likelihood $\left(\Theta_i,\mathrm{P_i},\ Cr_i\right)$ where $\Theta=\Theta''\times(\Theta_1\times\Theta_1')\times(\Theta_2\times\Theta_2')\times\ldots$, and where Cr is the measure of likelihood defined as shown below [4,5]

$$Cr(A)=\begin{cases} \sup\limits_{(\theta_1,\theta_2,\ldots)\in A} \inf\limits_{1\le k\le\infty} Cr_k\{\theta_k\}, & if \sup\limits_{(\theta_1,\theta_2,\ldots)\in A} \inf\limits_{1\le k\le\infty} Cr_k\{\theta_k\}<\frac{1}{2}, \\ 1-\sup\limits_{(\theta_1,\theta_2,\ldots)\in A^c} \inf\limits_{1\le k\le\infty} Cr_k\{\theta_k\}, & if \sup\limits_{(\theta_1,\theta_2,\ldots)\in A} \inf\limits_{1\le k\le\infty} Cr_k\{\theta_k\}\ge\frac{1}{2} \end{cases}$$

For every fixed θ, the quantities $M\chi_i(\theta)$ $M\gamma_i(\theta)$, $M\nu(\theta)$ and $M\omega(\theta)$ represent expectations of random variables $\chi(\theta)$ $\gamma(\theta)$, $\nu(\theta)$ and $\omega(\theta)$ respectively. As θ varies on the set Θ, then we already consider fuzzy quantities $M\chi_i(\theta)$ $M\gamma_i(\theta)$ $M\nu(\theta)$, $M\omega(\theta)$. For measurement of a fuzzy quantity, two critical values (optimistic and pessimistic values) with the specified confidential level α are used [5]. Thus, we can consider the following α-pessimistic and α-optimistic values of these expectations:

$$(M\chi_i(\theta))_{\sup}(\alpha)=\sup\left\{r\middle|Cr\left\{M\chi_i(\theta)\ge r\right\}\ge\alpha\right\},$$

$$(M\chi_i(\theta))_{\inf}(\alpha)=\inf\left\{r\middle|Cr\left\{M\chi_i(\theta)\le r\right\}\ge\alpha\right\},$$

$$(M\gamma_i(\theta))_{\sup}(\alpha)=\sup\left\{r\middle|Cr\left\{M\gamma_i(\theta)\ge r\right\}\ge\alpha\right\},$$

$$(M\gamma_i(\theta))_{\inf}(\alpha)=\inf\left\{r\middle|Cr\left\{M\gamma_i(\theta)\le r\right\}\ge\alpha\right\},$$

$$(M\nu(\theta))_{\sup}(\alpha)=\sup\left\{r\middle|Cr\left\{M\nu(\theta)\ge r\right\}\ge\alpha\right\},$$

$$(M\nu(\theta))_{\inf}(\alpha)=\inf\left\{r\middle|Cr\left\{M\nu(\theta)\le r\right\}\ge\alpha\right\},$$

$$(M\omega(\theta))_{\sup}(\alpha)=\sup\left\{r\middle|Cr\left\{M\omega(\theta)\ge r\right\}\ge\alpha\right\},$$

$$(M\omega(\theta))_{\inf}(\alpha)=\inf\left\{r\middle|Cr\left\{M\omega(\theta)\le r\right\}\ge\alpha\right\}.$$

For each $\theta\in\Theta$, the following equality is true

$$M\omega(\theta)=M\left(\sum_{i=1}^{\nu(\theta)-1}(\chi_i(\theta)+\gamma_i(\theta))+\chi_{\nu(\theta)}(\theta)\right).$$

Using the formula of total probability and definition of critical values α, we shall calculate α-pessimistic value $M\omega(\theta)$. Then

$$(M\omega(\theta))_{\inf}(\alpha) = M\left(\sum_{i=1}^{\nu(\cdot)-1}(\chi_i(\theta)+\gamma_i(\theta))+\chi_{\nu(\theta)}(\theta)\right)_{\inf}(\alpha) =$$
$$= \sum_{k=1}^{\infty}\left(P(\theta)=k)\sum_{i=1}^{k-1}(M\chi_i(\theta)+M\gamma_i(\theta))+M\chi_k(\theta)\right)_{\inf}(\alpha) =$$
$$= \sum_{k=1}^{\infty}P(\nu(\theta)=k)_{\inf}(\alpha)\left(\sum_{i=1}^{k-1}(M\chi_i(\theta)+M\gamma_i(\theta))+M\chi_k(\theta)\right)_{\inf}(\alpha) =$$
$$= \sum_{k=1}^{\infty}P(\nu(\theta=k)_{\inf}(\alpha)\left(\sum_{i=1}^{k-1}\left((M\chi_i(\theta))_{\inf}(\alpha)+(M\gamma_i(\theta))_{\inf}(\alpha)\right)+(M\chi_k(\theta))_{\inf}(\alpha)\right) =$$
$$= \left(M\chi(\theta)\sum_{k=1}^{\infty}P(\nu(\theta)=k)\right)_{\inf}(\alpha)+\left((M\chi(\theta)+M\gamma(\theta))\sum_{k=1}^{\infty}(k-1)P(\nu(\theta)=k)\right)_{\inf}(\alpha) =$$
$$= \left((M\chi(\theta))_{\inf}(\alpha)+(M\gamma(\theta))_{\inf}(\alpha)\right)(M\nu(\theta)-1)_{\inf}(\alpha)+(M\chi(\theta))_{\inf}(\alpha).$$

It can be similarly shown that α-optimistic value $M\omega(\theta)$ is defined by the following ratio

$$(M\omega(\theta))_{\sup}(\alpha) = \left((M\chi(\theta))_{\sup}(\alpha)+(M\gamma(\theta))_{\sup}(\alpha)\right)(M\nu(\theta)-1)_{\sup}(\alpha)+(M\chi(\theta))_{\sup}(\alpha).$$

Let us calculate now the average expected value of a random fuzzy quantity which is defined by the formula $E[\xi]=\frac{1}{2}\int_0^1\left(\xi_{\sup}(\alpha)+\xi_{\inf}(\alpha)\right)d\alpha$ [5,6], and we shall obtain the following

$$E[M\omega(\theta)] = \frac{1}{2}\int_0^1((M\chi)_{\inf}(\alpha)(M\nu(\theta))_{\inf}(\alpha)+(M\gamma(\theta))_{\inf}(\alpha)(M\nu(\theta)-1)_{\inf}(\alpha))d\alpha +$$
$$+\frac{1}{2}\int_0^1((M\chi(\theta))_{\sup}(\alpha)(M\nu(\theta))_{\sup}(\alpha)+(M\gamma(\theta))_{\sup}(\alpha)(M\nu(\theta)-1)_{\sup}(\alpha))d\alpha =$$
$$=\frac{1}{2}\int_0^1((M\chi(\theta))_{\inf}(\alpha)(M\nu(\theta))_{\inf}(\alpha)+(M\chi(\theta))_{\sup}(\alpha)(M\nu(\theta))_{\sup}(\alpha))d\alpha +$$
$$+\frac{1}{2}\int_0^1((M\gamma(\theta))_{\inf}(\alpha)(M\nu(\theta)-1)_{\inf}(\alpha)+(M\gamma(\theta))_{\sup}(\alpha)(M\nu(\theta)-1)_{\sup}(\alpha))d\alpha.$$

Thus, we obtain

$$E[M\omega(\theta)] = M\chi(\theta)+M\chi(\theta)(\nu(\theta)-1)+M\gamma(\theta)(\nu(\theta)-1). \quad (1)$$

From the definition of the average expected value of a random fuzzy quantity it follows that $M[\omega]=E[M\omega]$.

Practical use of the theory of fuzzy quantities assumes presence of membership functions. At the further calculation of the expectation of random fuzzy quantity $M[\omega]$, it is necessary to determine a membership function.

Let the mean time between failures of a facility of protection have an exponential distribution which parameter is a triangular fuzzy quantity. Then the mean time between failures of a facility of protection is a random fuzzy quantity. In addition to that, we shall assume that recovery time of a facility of protection has also an exponential distribution which parameter is a triangular fuzzy quantity. We believe that the number of a cycle ν on which there was a failure, also is a casual fuzzy quantity, and the probability of a failure on one cycle of operation process of a complex is equal to q.

Thus, proceeding from the assumptions made above, we shall write down:

$$P\{\chi(\theta)\le t\}=\begin{cases}1-e^{-\lambda_\chi(\theta)t}, & if \quad 0\le t<\infty,\\ 0, & if \quad t<0.\end{cases}$$

Let us note that a triangular fuzzy quantity $\lambda_\chi(\theta)$ can be represented by a group of three discrete numbers $\left(\lambda_\chi^{(1)},\lambda_\chi^{(2)},\lambda_\chi^{(3)}\right)$ in such a way that $\lambda_\chi^{(1)}<\lambda_\chi^{(2)}<\lambda_\chi^{(3)}$, and its membership function is defined by expression of the following form

$$\mu_{\lambda_\chi}(y_1)=\begin{cases}\dfrac{y_1-\lambda_\chi^{(1)}}{\lambda_\chi^{(2)}-\lambda_\chi^{(1)}}, & if \quad y_1\in\left[\lambda_\chi^{(1)},\lambda_\chi^{(2)}\right],\\ \dfrac{y_1-\lambda_\chi^{(3)}}{\lambda_\chi^{(2)}-\lambda_\chi^{(3)}}, & if \quad y_1\in\left[\lambda_\chi^{(2)},\lambda_\chi^{(3)}\right],\\ 0, \quad if \quad y_1\notin\left[\lambda_\chi^{(1)},\lambda_\chi^{(3)}\right].\end{cases}$$

Let us similarly write down the expression for a fuzzy quantity $\lambda_\chi(\theta)$

$$P\{\gamma(\theta)\le t\}=\begin{cases}1-e^{-\lambda_\gamma(\theta)t}, & if \quad 0\le t<\infty,\\ 0, & if \quad t<0\end{cases}$$

$$\mu_{\lambda_\gamma}(y_2)=\begin{cases}\dfrac{y_2-\lambda_\gamma^{(1)}}{\lambda_\gamma^{(2)}-\lambda_\gamma^{(1)}}, & if \quad y_2\in\left[\lambda_\gamma^{(1)},\lambda_\gamma^{(2)}\right],\\ \dfrac{y_2-\lambda_\gamma^{(3)}}{\lambda_\gamma^{(2)}-\lambda_\gamma^{(3)}}, & if \quad y_2\in\left[\lambda_\gamma^{(2)},\lambda_\gamma^{(3)}\right],\\ 0, \quad if \quad y_2\notin\left[\lambda_\gamma^{(1)},\lambda_\gamma^{(3)}\right].\end{cases}$$

As the considered process of protected facility operation is an alternating random process, then the probability of the failure occurring on the k-th cycle of regeneration is equal as shown below

$$P\{\nu = k\} = (1-q)^{(k-1)} q. \qquad (2)$$

In this case, it is not possible to explicitly write the expression for a membership function of the quantity $M\omega$, however it is possible to calculate $E[\omega]$. It is obvious that

$$M[\omega] = M\left[\frac{1}{q\lambda_\chi} + \frac{1-q}{q\lambda_\gamma}\right] = \frac{1}{q} M\left[\frac{1}{\lambda_\chi}\right] + \frac{1-q}{q} M\left[\frac{1}{\lambda_\gamma}\right].$$

In the obtained expression, there were unknown expected values $M\left[\frac{1}{\lambda_\chi}\right]$ and $M\left[\frac{1}{\lambda_\gamma}\right]$ which should be calculated. The expectation of a fuzzy quantity $M\left[\frac{1}{\lambda_\chi}\right]$ shall be calculated as follows:

$$M\left[\frac{1}{\lambda_\chi}\right] = \int_0^{+\infty} Cr\left\{\theta \in \Theta \middle/ \frac{1}{\lambda_\chi(\theta)} \geq r \right\} dr.$$

Before to calculate $M\left[\frac{1}{\lambda_\chi}\right]$, first it is necessary to define the corresponding membership function:

$$\mu(y) = \sup_{y=\frac{1}{y_1}} \mu_{\lambda_\chi}(y_1) = \begin{cases} \dfrac{\frac{1}{y_1} - \lambda_\chi^{(1)}}{\lambda_\chi^{(2)} - \lambda_\chi^{(1)}}, & if \quad y \in \left[\frac{1}{\lambda_\chi^{(2)}}, \frac{1}{\lambda_\chi^{(1)}}\right] \\ \dfrac{\frac{1}{y_1} - \lambda_\chi^{(3)}}{\lambda_\chi^{(2)} - \lambda_\chi^{(3)}}, & if \quad y \in \left[\frac{1}{\lambda_\chi^{(3)}}, \frac{1}{\lambda_\chi^{(2)}}\right] \\ 0, & if \quad y \notin \left[\frac{1}{\lambda_\chi^{(3)}}, \frac{1}{\lambda_\chi^{(1)}}\right] \end{cases}.$$

Then the measure of likelihood can be written down in the following form

$$Cr\left\{\theta\in\Theta \Big/ \frac{1}{\lambda_\chi(\theta)}\geq r\right\}=\begin{cases} 1, & if \quad r\leq\dfrac{1}{\lambda_\chi^{(3)}} \\ 1-\dfrac{1}{2}\dfrac{\frac{1}{r}-\lambda_\chi^{(3)}}{\lambda_\chi^{(2)}-\lambda_\chi^{(3)}}, & if \quad r\in\left[\dfrac{1}{\lambda_\chi^{(3)}},\dfrac{1}{\lambda_\chi^{(2)}}\right] \\ \dfrac{1}{2}\dfrac{\frac{1}{r}-\lambda_\chi^{(1)}}{\lambda_\chi^{(2)}-\lambda_\chi^{(1)}}, & if \quad r\in\left[\dfrac{1}{\lambda_\chi^{(2)}},\dfrac{1}{\lambda_\chi^{(1)}}\right] \\ 0, & if \quad r>\dfrac{1}{\lambda_\chi^{(1)}} \end{cases}.$$

Consequently, we obtain

$$M\left[\frac{1}{\lambda_\chi}\right]=\int_0^{+\infty}Cr\left\{\theta\in\Theta \Big/ \frac{1}{\lambda_\chi(\theta)}\geq r\right\}dr=\frac{1}{2}\,\frac{1}{\lambda_\chi^{(2)}-\lambda_\chi^{(1)}}\ln\left(\frac{\lambda_\chi^{(2)}}{\lambda_\chi^{(1)}}\right)+\frac{1}{2}\,\frac{1}{\lambda_\chi^{(3)}-\lambda_\chi^{(2)}}\ln\left(\frac{\lambda_\chi^{(3)}}{\lambda_\chi^{(2)}}\right). \qquad (3)$$

The expectation of a fuzzy quantity $M\left[\frac{1}{\lambda_\gamma}\right]$ can be written down similarly to (3), that is

$$M\left[\frac{1}{\lambda_\gamma}\right]=\frac{1}{2}\,\frac{1}{\lambda_\gamma^{(2)}-\lambda_\gamma^{(1)}}\ln\left(\frac{\lambda_\gamma^{(2)}}{\lambda_\gamma^{(1)}}\right)+\frac{1}{2}\,\frac{1}{\lambda_\gamma^{(3)}-\lambda_\gamma^{(2)}}\ln\left(\frac{\lambda_\gamma^{(3)}}{\lambda_\gamma^{(2)}}\right).$$

Let us now define $M\nu$. As ν is a discrete random variable, then

$$M\nu=\sum_{k=1}^{\infty}kq(1-q)^{k-1}=q\,.$$

We substitute the obtained expressions $M\left[\frac{1}{\lambda_\gamma}\right]$, $M\left[\frac{1}{\lambda_\chi}\right]$ and $M\nu$ in the formula $M[\omega]$ and obtain the following

$$\begin{aligned} M[\omega]=&\frac{1}{2q}\left\{\frac{1}{\lambda_\chi^{(2)}-\lambda_\chi^{(1)}}\ln\left(\frac{\lambda_\chi^{(2)}}{\lambda_\chi^{(1)}}\right)+\frac{1}{\lambda_\chi^{(3)}-\lambda_\chi^{(2)}}\ln\left(\frac{\lambda_\chi^{(3)}}{\lambda_\chi^{(2)}}\right)\right\}+ \\ &+\frac{1-q}{2q}\left\{\frac{1}{\lambda_\gamma^{(2)}-\lambda_\gamma^{(1)}}\ln\left(\frac{\lambda_\gamma^{(2)}}{\lambda_\gamma^{(1)}}\right)+\frac{1}{\lambda_\gamma^{(3)}-\lambda_\gamma^{(2)}}\ln\left(\frac{\lambda_\gamma^{(3)}}{\lambda_\gamma^{(2)}}\right)\right\} \end{aligned}.$$

It should be noted that if $\lambda_\chi^{(1)} = \lambda_\chi^{(2)} = \lambda_\chi^{(3)}$ and $\lambda_\gamma^{(1)} = \lambda_\gamma^{(2)} = \lambda_\gamma^{(3)}$, the obtained expression is reduced to expression for calculation of an expectation in case when parameters of distributions are discrete quantities.

To calculate $M\omega(\theta)$, as mathematical expectation of a fuzzy quantity, it is necessary to obtain a ratio for the probability $q(\theta)$ at each fixed $\theta \in \Theta$ since values $M\chi(\theta)$ and $M\gamma(\theta)$ have been calculated above. Let us consider the operation process of a safety system in more detail. As a safety system operates in a protected facility failure expectation mode of expectation, it is impossible to detect its failures during the moment of their occurrence. Therefore, for detection of latent failures, the procedure of the periodic preventive control of a safety system is introduced. We shall designate the period of the control of a safety system's operability as T, and its duration as δ. We assume that during the periodic control, a safety system stops to carry out its functions. If a safety system has regularly operated over the time ξ_i on the i-th cycle of regeneration, then for this period of time there has been made $\left[\frac{\xi_i}{T+\delta}\right]$ ($[x]$, $\{x\}$ are the integer and the fractional part of the number x, $x^+ = \max(x,0)$ respectively) periods of control preventive maintenance. And for this time, the system has been in upstate for the time equal to $\left[\frac{\xi_i}{T+\delta}\right]T$. A failure F is detected only during the moments of time $\left(\left[\frac{\xi_i}{T+\delta}\right]+1\right)(T+\delta)$, i.e. during the moments of control procedure termination. After carrying out corresponding repair-and-renewal operations with the duration η_i a safety system starts again to function regularly. Thus, the process of safety system functioning with control preventive maintenance is a random process with the period of regeneration

$$\tau_F(\xi,\eta) = (T+\delta)\left(\left[\frac{\xi}{T+\delta}\right]+1\right)+\eta .$$

Thus, the failure of a complex occurs in case when a protected facility's failure falls down on a downstate of a safety system. The moments of regeneration of a complex's operation process will be the moments of the termination of protected facility recovery.

Considering the independence of examined quantities, the ratio (1) can be written down for each fixed $\theta \in \Theta$ in the following form

$$\begin{aligned} E[\omega(\theta)] &= \frac{1}{q(\theta)} M\chi(\theta) + \frac{1-q(\theta)}{q(\theta)} M\gamma(\theta) = \\ &= \frac{1}{q(\theta)} \int_0^\infty \left(1 - F_\chi\left(t;\vec{\lambda}_\chi(\theta)\right)\right) dt + \frac{1-q(\theta)}{q(\theta)} \int_0^\infty \left(1 - F_\gamma\left(t;\vec{\lambda}_\gamma(\theta)\right)\right) dt, \end{aligned} \tag{4}$$

where $q(\theta)$ is the probability of a failure on a cycle of a complex's operation process regeneration $F_\chi\left(t;\vec{\lambda}(\theta)\right)$, $F_\gamma\left(t;\vec{\lambda}_\gamma(\theta)\right)$ are probabilistic distributions for random fuzzy variables χ and γ respectively.

For calculation of the probability $q(\theta)$, we shall designate the set of the moments of time during which the safety system is capable to parry failures of a facility under protection as $Q^+(\theta)$ and the set of the moments of time during which the safety system is not capable to parry failures of a facility under protection as $Q^-(\theta)$. Then we shall calculate the probability $q(\theta)$ for each fixed $\theta \in \Theta$ as follows:

$$q(\theta)=\int_0^{\infty}P\left(t\in Q^-(\theta)\right)dF_\chi\left(t;\vec{\lambda}_\chi(\theta)\right).$$

Noticing that

$$P\left(t\in Q^-(\theta)\right)=1-P\left(t\in Q^+(\theta)\right)=1-P^+(t;\theta).$$

Then, according to the formula of total probability, we shall write down

$$P^+(t;\theta)=\int_0^{\infty}\int_0^{\infty}P\left(t\in Q^+(\theta)\,|\,\xi(\theta)=x,\eta(\theta)=y\right)dF_\eta\left(y;\vec{\lambda}_\eta(\theta)\right)dF_\xi\left(x;\vec{\lambda}_\xi(\theta)\right).$$

Let us remind that ξ is a random operating time to the first latent failure of a safety system, and η is the random time of a safety system's recovery after the first latent failure. For calculation of $P^+(t;\theta)$, there are only two possible mutually exclusive alternatives: $\tau_{СБ}(\xi,\eta)>t$ and $\tau_F(\xi,\eta)\le t$. Therefore, we shall present $P^+(t)$ in the form of the sum of two summands [2]:

$$P^+(t;\theta)=\iint\limits_{\tau_{СБ}(x,y)\le t}P\left(t\in Q^+(\theta)\,|\,\xi(\theta)=x,\eta(\theta)=y\right)dF_\eta\left(y;\vec{\lambda}_\eta(\theta)\right)dF_\xi\left(x;\vec{\lambda}_\xi(\theta)\right)+$$
$$+\iint\limits_{\tau_{СБ}(x,y)>t}P\left(t\in Q^+(\theta)\,|\,\xi(\theta)=x,\eta(\theta)=y\right)dF_\eta\left(y;\vec{\lambda}_\eta(\theta)\right)dF_\xi\left(x;\vec{\lambda}_\xi(\theta)\right)=I_1+I_2.$$

At first, let us calculate the second summand I_2. The condition $\tau_F(x,y)>t$ means that the moment of regeneration of a safety system's operation process has come after the moment of time t. Taking into account this condition, we shall transform I_2 as follows:

$$I_2=\iint\limits_{\tau_{СБ}(x,y)>t}\left(\sum_{m=0}^{\left[\frac{x}{T+\delta}\right]-1}J_{t\in[m(T+\delta,m(T+\delta)+T)}+J_{t\in\left[\left[\frac{x}{T+\delta}\right](T+\delta),x\right)}\right)dF_\eta\left(y;\vec{\lambda}_\eta(\theta)\right)dF_\xi\left(x;\vec{\lambda}_\eta(\theta)\right)=$$
$$=\int_0^{\infty}\left(\sum_{m=0}^{\left[\frac{x}{T+\delta}\right]-1}J_{t\in[m(T+\delta,m(T+\delta)+T)}+J_{t\in\left[\left[\frac{x}{T+\delta}\right](T+\delta),x\right)}\right)\left(1-F_\eta\left(t-\left(\left[\frac{x}{T+\delta}\right]+1\right)(T+\delta);\vec{\lambda}_\eta(\theta)\right)\right)dF_\xi\left(x;\vec{\lambda}_\eta(\theta)\right),$$

where $J_{t\in A}$ is the indicator of event $t\in A$.

Omitting some simple but bulky transformations, we shall give the result at once:

$$I_2 = \left(1 - F_\xi\left(t; \vec{\lambda}_\xi(\theta)\right)\right) - \sum_{m=1}^{\infty}\left(1 - F_\xi\left(m(T+\delta); \vec{\lambda}_\xi(\theta)\right)\right)\left(J_{(m-1)(T+\delta)+T \le t} - J_{m(T+\delta) \le t}\right).$$

It should be noted that $1 - \sum_{m=1}^{\infty}(1 - F_\xi(m(T+\theta)))(J_{(m-1)(T+\theta)\le t} - J_{m(T+\theta)\le t})$ represents the distribution function $F_\zeta(t)$ of some random variable ζ that allows one to present I_2 in the following form

$$I_2 = F_\zeta\left(t; \vec{\lambda}_\xi(\theta)\right) - F_\xi\left(t; \vec{\lambda}_\xi(\theta)\right). \tag{5}$$

Now let us consider I_1. When considering the case $\tau_F \le t$, we shall take into account that after carrying out corresponding repair-and-renewal operations during the moment of time τ_F, a safety system will again function with the same probabilistic characteristics, as during the moment of time $t = 0$. Then

$$I_1 = \iint_{\tau_{CБ}(x,y)\le t} P^+(t - \tau_{CБ}(x,y))dF_\eta\left(y; \vec{\lambda}_\eta(\theta)\right)dF_\xi\left(x; \vec{\lambda}_\xi(\theta)\right) = \int_0^t P^+(t-z)dF_{\tau_{CБ}}\left(z; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right). \tag{6}$$

Thus, by summarizing (5) and (6), for each fixed $\theta \in \Theta$, we obtain the Volterra type integrated equation of the second kind for the probability $P\left(t \in Q^+(\theta)\right)$:

$$P^+(t;\theta) = f(t;\theta) + \int_0^t P^+(t-z)dF_{\tau_F}\left(z; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right), \tag{7}$$

where $f(t;\theta) = F_\zeta\left(t; \vec{\lambda}_\xi(\theta)\right) - F_\xi\left(t; \vec{\lambda}_\xi(\theta)\right)$,

$$F_{\tau_F}\left(z; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right) = P\left(\left(\left[\frac{\xi(\theta)}{T+\delta}\right] + 1\right)(T+\delta) + \eta(\theta) \le z\right).$$

As equation (7) is a convolution equation, then for its solution we use the Laplace-Stieltjes transform. The solution of equation (7) in terms of the Laplace-Stieltjes transform has the following form

$$\tilde{P}^+(s;\theta) = \frac{\tilde{f}(s;\theta)}{1 - \tilde{F}_{\tau_F}\left(s; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right)} = \frac{Me^{-s\zeta(\theta)} - Me^{-s\xi(\theta)}}{1 - Me^{-s\tau_F(\theta)}},$$

where $\tilde{P}^+(s;\theta) = \int_0^\infty e^{-st}dP^+(t;\theta)$, $\tilde{F}_{\tau_F}\left(s; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right) = \int_0^\infty e^{-st}dF_{\tau_F}\left(t; \vec{\lambda}_\eta(\theta), \vec{\lambda}_\xi(\theta)\right)$ and $\tilde{f}(s;\theta) = \int_0^\infty e^{-st}df(t;\theta) = \int_0^\infty e^{-st}dF_\zeta\left(t; \vec{\lambda}_\xi(\theta)\right) - \int_0^\infty e^{-st}dF_\xi\left(t; \vec{\lambda}_\xi(\theta)\right)$.

Due to the difficulty of finding a reverse conversion of the Laplace-Stieltjes transform in the analytical form, it is not possible, so we shall find an asymptotic ratio for $P^{+}(t;\theta)$ applying a known ratio $\lim_{t\to\infty} P^{+}(t;\theta)=\lim_{s\to 0}\tilde{P}^{+}(s;\theta)$. Applying the specified ratio, we obtain

$$\lim_{t\to\infty} P^{+}(t;\theta)=\frac{M\xi(\theta)-M\zeta(\theta)}{M\tau_F(\theta)}.$$

It is obvious that now it is necessary to calculate expressions for expectations $M\zeta(\theta)$, $M\xi(\theta)$ and $M\tau_F(\theta)$. And $M\zeta(\theta)=\int_0^\infty\left(1-F_\zeta\left(t;\vec{\lambda}_\xi(\theta)\right)\right)dt$, then

$$M\zeta(\theta)=\int_0^\infty\sum_{m=1}^{\infty}(1-F_\xi(k(T+\delta);\vec{\lambda}_\xi(\theta)))(J_{(m+1)(T+\delta)+T\le t}-J_{m(T+\delta)\le t})dt=$$
$$=\sum_{k=0}^{\infty}k\delta\left(F_\xi\left((k+1)(T+\delta);\vec{\lambda}_\xi(\theta)\right)-F_\xi\left(k(T+\delta);\vec{\lambda}_\xi(\theta)\right)\right)=\sum_{k=0}^{\infty}k\delta P\left(\left[\frac{\xi(\theta)}{T+\delta}\right]=k\right)=\delta\, M\left[\frac{\xi(\theta)}{T+\delta}\right].$$

As $\xi(\theta)>>T+\delta$, $\left[\frac{\xi(\theta)}{T+\delta}\right]\cong\frac{\xi(\theta)}{T+\delta}-\frac{1}{2}$ [1] that allows to rewrite $M\zeta(\theta)$ for each fixed $\theta\in\Theta$. Therefore, we have $M\zeta(\theta)=M\xi(\theta)\frac{\delta}{T+\delta}-\frac{\delta}{2}$. Then

$$M\xi(\theta)-M\zeta(\theta)=M\xi(\theta)\frac{T}{T+\delta}+\frac{\delta}{2}\text{ And }M\tau_F(\theta)=M\eta(\theta)+\frac{T+\delta}{2}+M\xi(\theta).$$

$$\lim_{t\to\infty} P^{+}(t;\theta)=\frac{M\xi(\theta)\frac{T}{T+\delta}+\frac{\delta}{2}}{M\xi(\theta)+\frac{T+\delta}{2}+M\eta(\theta)}.$$

It should be noted that $M\eta(\theta)\,M\xi(\theta)$ can be calculated similarly to expectations $M\chi(\theta)$ and $M\gamma(\theta)$. For the probability $q(\theta)$ for each fixed $\theta\in\Theta$, we can write down the following asymptotic estimation:

$$q(\theta)\approx 1-\lim_{t\to\infty} P^{+}(t;\theta)=\frac{M\xi(\theta)\frac{\delta}{T+\delta}+\frac{T}{2}+M\eta(\theta)}{M\xi(\theta)+\frac{T+\delta}{2}+M\eta(\theta)}.$$

Thus, we have obtained the ratio for all quantities in equation (1). Now we can calculate $M\left[\omega(\theta)\right]$ for each fixed $\theta\in\Theta$ under the formula

$$M\left[\omega(\theta)\right]=\frac{\left(M\xi(\theta)+\dfrac{T+\delta}{2}+M\eta(\theta)\right)M\chi(\theta)+\left(M\xi(\theta)\dfrac{T}{T+\delta}+\dfrac{\delta}{2}\right)M\gamma(\theta)}{M\xi(\theta)\dfrac{\delta}{T+\delta}+\dfrac{T}{2}+M\eta(\theta)},$$

where $M\chi(\theta)=\int\limits_0^\infty\left(1-F_\chi\left(t;\vec{\lambda}_\chi(\theta)\right)\right)dt$, $M\xi(\theta)=\int\limits_0^\infty\left(1-F_\xi\left(t;\vec{\lambda}_\xi(\theta)\right)\right)dt$, $M\gamma(\theta)=\int\limits_0^\infty\left(1-F_\gamma\left(t;\vec{\lambda}_\gamma(\theta)\right)\right)dt$

and $M\eta(\theta)=\int\limits_0^\infty\left(1-F_\eta\left(t;\vec{\lambda}_\eta(\theta)\right)\right)dt$.

Consequently, according to the definition of fuzzy values function [4,5], we have given a fuzzy expectation of mean operating time to the first failure as function of model fuzzy parameters.

Now we shall take advantage of Zadeh’s extension principle [6]:

$$\mu(x)=\sup_{x=f(x_1,x_2,\ldots,x_n)}\min_{1\le i\le n}\mu_i(x_i)$$

and write down a ratio for the membership function $M\omega(\theta)$:

$$\mu_{M\omega}(y)=\sup_{y=f(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)}\min\left(\mu_{\vec{\lambda}_\chi}(\vec{x}_1),\mu_{\vec{\lambda}_\gamma}(\vec{x}_2),\mu_{\vec{\lambda}_\xi}(\vec{x}_3),\mu_{\vec{\lambda}_\eta}(\vec{x}_4)\right),$$

Where

$$f(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4)=\frac{\left(M\xi(\vec{x}_3)+\dfrac{T+\delta}{2}+M\eta(\vec{x}_4)\right)M\chi(\vec{x}_1)+\left(M\xi(\vec{x}_3)\dfrac{T}{T+\delta}+\dfrac{\delta}{2}\right)M\gamma(\vec{x}_2)}{M\xi(\vec{x}_3)\dfrac{\delta}{T+\delta}+\dfrac{T}{2}+M\eta(\vec{x}_4)}.$$

Thus, we have managed to write down a ratio for membership function of the expected value of a complex’s operating time to failure through membership functions of model parameters.

Now let us consider the defuzzification procedure, i.e. the transformation of a fuzzy set into a discrete quantity. For this purpose, we shall take advantage of the definition of the average expected value of random fuzzy variables [6]:

$$M\left[\omega(\theta)\right]=\int\limits_0^\infty Cr\{\theta\in\Theta\,|\,M\omega(\theta)\ge r\}dr-\int\limits_{-\infty}^0 Cr\{\theta\in\Theta\,|\,M\omega(\theta)\le r\}dr.$$

Let us find a corresponding measure of likelihood using a ratio connecting a measure of likelihood and a membership function [5]:

$$Cr\{M\omega \in B\} = \frac{1}{2}\left(\sup_{y\in B}\mu_{M\omega}(y) + 1 - \sup_{y\in R\setminus B}\mu_{M\omega}(y)\right).$$

Then we shall write down

$$M[\omega] = \frac{1}{2}\int_0^\infty \left(\sup_{y\geq r}\mu_{M\omega}(y) + 1 - \sup_{y<r}\mu_{M\omega}(y)\right)dr.$$

Assuming that random fuzzy quantities $\xi(\theta)$ and $\eta(\theta)$ are distributed according to the exponential law which parameters are triangular fuzzy quantities, we can see that expected values $M[\xi(\theta)]$ and $M[\eta(\theta)]$ can be written down similarly to ratio (3).

An example

Let random fuzzy quantities χ ξ, γ and η be distributed according to the exponential law; i.e.

$F_\chi\{t;\lambda(\theta)\} = 1 - e^{-\lambda_\chi(\theta)t}$, $F_\xi\{t;\lambda(\theta)\} = 1 - e^{-\lambda_\xi(\theta)t}$, $F_\gamma\{t;\lambda(\theta)\} = 1 - e^{-\lambda_\lambda(\theta)t}$ and $F_\eta\{t;\lambda(\theta)\} = 1 - e^{-\lambda_\eta(\theta)t}$.

Here we shall construct a membership function based on distribution parameters’ confidential intervals using the method offered by J. Buckley [8]. Its essence consists in the fact that the membership function of a required distribution parameter is defined by its sets of α-level. In this case the interval estimation of the required parameter of distribution with a confidence level $(1-\alpha)$ is taken as a set of α-level. Confidence limits for a failure rate are calculated under the following formulas [12]:

$$\lambda_{i} = \frac{\chi^2(1-\alpha_1, 2d)}{2nt_0}\ \lambda_{\hat{a}} = \frac{\chi^2(\alpha_2, 2d)}{2nt_0},$$

where d is the number of failures over the time t_0, n is the total number of elements of the given name, t_0 is the time of operation (in hours). In this case t_0=289080 h, λ_i is the lower limit of the confidential interval, $\lambda_{\hat{a}}$ is the upper limit of the confidential interval, α_1 is the probability of the event $\lambda \geq \lambda_i$, α_2 is the probability of the event $\lambda \leq \lambda_{\hat{a}}$, $\alpha = \alpha_1 + \alpha_2 - 1$ is the probability of event $\lambda_i \leq \lambda \leq \lambda_{\hat{a}}$, $\chi^2(s,r)$ is the quantile of χ^2 distribution with a parameter s and the number of degrees of freedom r.

In this case, the membership function of the expected value of a complex operating time before failure we shall write down as follows:

$$\mu_{M\omega}(y) = \sup_{y=f(x_1,x_2,x_3,x_4)}\min\left(\mu_{\lambda_\chi}(x_1), \mu_{\lambda_\gamma}(x_2), \mu_{\lambda_\xi}(x_3), \mu_{\lambda_\eta}(x_4)\right),$$

where $f(x_1,x_2,x_3,x_4) = \frac{1}{q(x_3,x_4)}\frac{1}{x_1} + \frac{1-q(x_3,x_4)}{q(x_3,x_4)}\frac{1}{x_2}$, $q(x_3,x_4) \approx \dfrac{\frac{1}{x_3}\frac{\delta}{T+\delta} + \frac{T}{2} + \frac{1}{x_4}}{\frac{1}{x_3} + \frac{T+\delta}{2} + \frac{1}{x_4}}$.

The elementary methods of defuzzification are: the method of the centre of gravity, the method of the center of areas, the method of the left modal value and the method of the right modal value [11].

In this case $\mu_{\lambda_\chi}(x) = \Delta\left(1\times10^{-6}h^{-1}, 1.5\times10^{-6}h^{-1}, 2\times10^{-6}h^{-1}\right)$, $\mu_{\lambda_\gamma}(x) = \Delta\left(1h^{-1}, 1.5h^{-1}, 2h^{-1}\right)$ $\mu_{\lambda_\chi}(x) = \Delta\left(1\times10^{-4}h^{-1}, 1.5\times10^{-4}h^{-1}, 2\times10^{-4}h^{-1}\right)$, $\mu_{\lambda_\eta}(x) = \Delta\left(1h^{-1}, 1.5h^{-1}, 2h^{-1}\right)$, $T = 500h$, $\delta = 0.1h$. Here Δ designates a triangular membership function. Then we have

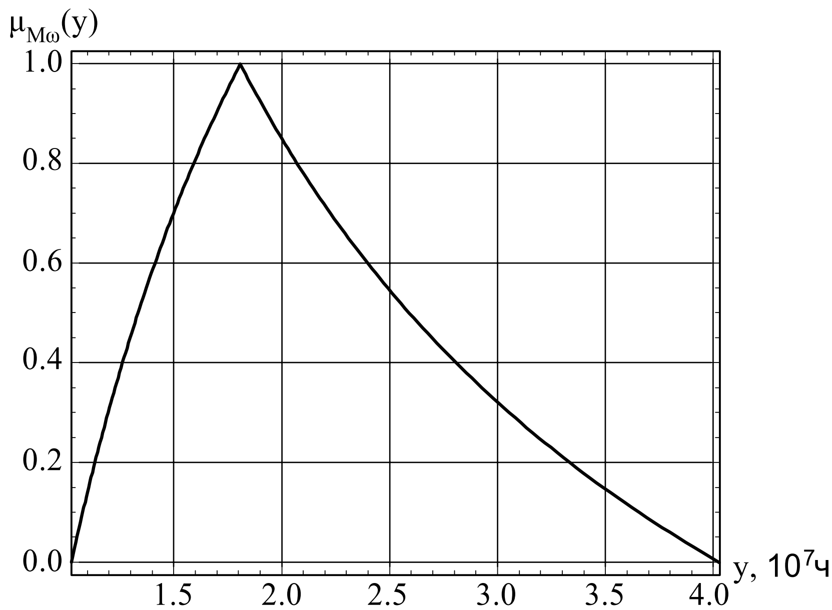

Fig. 1. Membership function of mean operating time to failure of a complex

At the same time, the value $M\omega$ calculated in the classical way makes $1.808\times10^{7}h$ that coincides with a maximum of membership function, and the value $M[\omega]$ obtained as a result of defuzzification procedure makes $2.029\times10^{7}h$ that reflects asymmetric property of obtained membership function. We also have estimated the contribution of each of fuzzy parameters to uncertainty of a result according to [8]. So, for the parameter λ_χ, we have obtained 0.506858, for λ_γ – 0.4×10^{-7}, for λ_ξ – 0.491839, and for λ_η – 0.00130296.

Conclusion

Thus, the given study has presented an approach to the estimation of mean operating time to failure of a complex "facility of protection – safety system" in view of uncertainty of initial data in the task. Considering the process of a complex's operation as a random fuzzy process of recovery, the ratios have been obtained that allow us to write down a membership function for mean operating time to failure of a complex. Knowing the membership function of a complex's parameters, mean operating time to failure of a complex has been estimated by applying defuzzification procedure.

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