Original article
https://doi.org/10.21683/1729-2646-2022-22-1-52-55

Synthesis of new, more powerful statistical tests through multiplicative clustering of classical Frozini and Murota-Takeuchi tests with the Hurst test for the purpose of testing small samples for normality

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Summary. Aim. *The paper examines the problem of small sample analysis by means of synthesizing new statistical tests generated by the clustering of the Hurst statistical test with the Frozini test, as well as with the Murota-Takeuchi test. The problem of normal distribution hypothesis testing on samples of 16 to 25 experiments is solved. Such significant limitations of the sample size arise in subject areas that include biometrics, biology, medical science and economics. In this case, the problem can be solved by applying not one, but a number of statistical tests to the analysis of the same small sample.* **Methods.** *It is suggested multiplying the Hurst test outputs by the Frozini test and/or the Murota-Takeuchi test outputs. A multiplicative clustering was performed for pairs of examined tests and their combination. It was shown that for each known statistical test, an equivalent artificial neuron can be constructed. A neural network integration of about 21 classical statistical tests constructed in the last century becomes possible. It is expected that the addition of new statistical tests in the form of artificial neurons will improve the quality of multi-criteria analysis solutions. Formally, the products of non-recurrent pairs of 21 original classical statistical tests should produce 210 new statistical tests. That is significantly more than the total number of statistical tests developed in the last century for the purpose of normality testing.* **Results.** *Pairwise product of the examined tests allows reducing the probability of errors of the first and second kind by more than 1.55 times as compared to the basic Hurst test. In case of triple product of the tests, the probabilities of error decrease relative to the basic Hurst test and to the associated second test. It is noted that there is no steady improvement in the quality of the decisions made by multiplicative mathematical constructions. The probabilities of error of the new test obtained by multiplying three of the examined tests are approximately 1.5% worse as compared to those of the tests obtained by multiplying pairs of the original tests.* **Conclusions.** *By analogy with the examined tests, the proposed data processing methods can also be applied to other known statistical tests. In theory, it becomes possible to significantly increase the number of new statistical tests by multiplying their final values. Unfortunately, as the number of clustered statistical tests grows, mutual correlations between the newly synthesized tests grow as well. The latter fact limits the capabilities of the method proposed in the paper. Further research is required in order to identify the most efficient combinations of pairs, triples or large groups for known statistical tests.*

Keywords: *statistical analysis of small samples, normality testing, Hurst test, Frozini test, Murota-Takeuchi test.*

For citation: *Ivanov A.I., Kupriyanov E.N. Synthesis of new, more powerful statistical tests through multiplicative clustering of classical Frozini and Murota-Takeuchi tests with the Hurst test for the purpose of testing small samples for normality. Dependability 2022;1: 52-55. https://doi.org/10.21683/1729-2646-2022-22-1-52-55*

Received on: *08.10.2021* / **Revised on:** *28.01.2022* / **For printing:** *18.03.2022*

Introduction

Neural networks are trained to convert biometrics into an authentication code per GOST R 52633.5 [1] using 16 examples of a "Friend" image. "Good" biometrics data have a normal distribution, while "bad" data with gross errors have a near-uniform distribution. Eventually, when evaluating the quality of small learning samples, the hypothesis of normal distribution of a small sample of 16 examples needs to be tested.

One of the methods of testing the normality hypothesis involves using the Hurst test (the ratio between the scope of data and the standard deviation of the sample that is commonly used in economics [2]). Unfortunately, this statistical test does not perform well with small samples. The distribution of data is shown in Fig. 1.

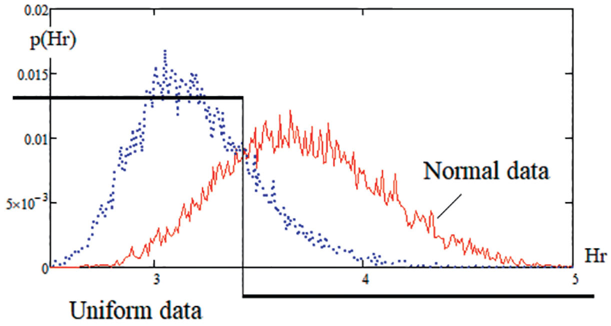

Fig. 1. Example of poor linear separability by an artificial neuron of the output states of the classical Hurst test for small samples of 16 experiments

It is obvious that, for small samples, the probabilities of errors of the first and second kind are high: $P_1 = P_2 = P_{EE} \approx 0.228$. In this context, according to the standard recommended [3, 4] acceptable values of the confidence probability, the classical tests are to be applied to samples of 200 or more experiments. This condition cannot be fulfilled for neural network biometrics.

A similar situation occurs when we try using another statistical test. Fig. 2 shows the density functions of the output states of the Frozini test.

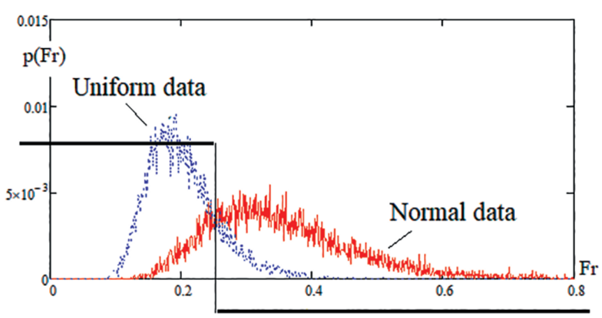

Fig. 2. Output states of the classical Frozini test for small samples of 16 experiments

Comparing Fig. 1 and Fig. 2 clearly shows that the linear separability of normal and uniform small sample data of the Frozini test is significantly better $P_1 = P_2 = P_{EE} \approx 0.172$ as compared to the Hurst test. We observe a 1.33-fold decrease in the probability of errors of the first and second kind. The effect of linear separability of data is even higher for the Murota-Takeuchi test, Fig. 3.

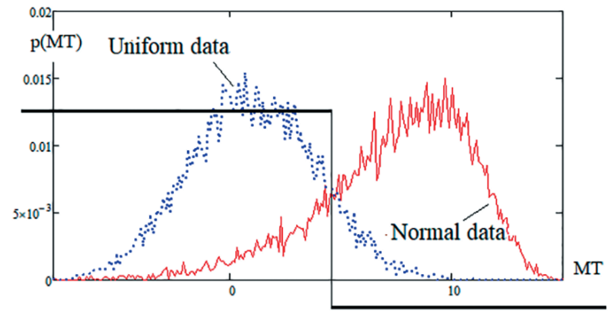

Fig. 3. Distribution of the output data of the Murota-Takeuchi test

For the Murota-Takeuchi test, the probability of errors of the first and second kind can be reduced to $P_1 = P_2 = P_{EE} \approx 0.152$. That test is the most powerful out of the three examined.

It is also obvious that for each statistical test [5, 6], an equivalent artificial neuron can be constructed that quantizes data in the point of equally probable errors of the first and second kind $P_1 = P_2 = P_{EE}$, if the quantizer outputs "0" for normal data. In this case, the three artificial neurons under consideration will likely output code "000" with triple redundancy if the input is data with a normal-like distribution.

Moreover, the reference book [7] describes 21 statistical tests for normality testing. In other words, we can obtain 21 artificial neurons that solve the same problem simultaneously. At the same time, formally, we will obtain output codes with a 21-fold redundancy. This redundancy can be contracted using codes that enable detection and correction of errors [8].

Unfortunately, most of the statistical tests created in the last century involve a strong correlation between the output states. Taking into account the effect of the correlations [9] causes the situation whereas the statistical tests created in the last century are not sufficient for a confidence probability of 0.99. About 40 new statistical tests need to be synthesised in the near future.

Method of increasing the number of statistical tests through pairwise multiplication of their final results

It should be noted that multiplying the outputs of the formula of a certain statistical test should cause increased linear separability of small samples with a normal and uniform distribution. This fact is easily verified through a numerical experiment. The Hurst, Frozini and Murota-Takeuchi tests, as well as their multiplicative clusterings are calculated using software written in MathCAD and shown Fig. 4.

$$
sx := \left| \begin{array}{l}
x \leftarrow \mathrm{sort}(\mathrm{rnorm}(16,0,1)) \\
m \leftarrow \mathrm{mean}(x) \\
\sigma \leftarrow \mathrm{stdev}(x) \\
Hr \leftarrow \dfrac{x_{15} - x_0}{\sigma} \\
Fr \leftarrow \displaystyle\sum_{i=0}^{15} \left[\left| \left(\mathrm{pnorm}(x_i, m, \sigma) - \frac{i-0.5}{16} \right) \right| \cdot \mathrm{dnorm}(x_i, m, \sigma) \right] \\
MT \leftarrow \displaystyle\sum_{i=0}^{15} \left(\cos\left(\frac{x_i - x_{15}}{1.8} \right) \right) \\
(Hr \;\; Fr \;\; MT \;\; Hr \cdot Fr \;\; Hr \cdot MT \;\; Hr \cdot Fr \cdot MT)^T
\end{array} \right.
\qquad
sxr := \left| \begin{array}{l}
x \leftarrow \mathrm{sort}(\mathrm{runif}(16,-3,3)) \\
m \leftarrow \mathrm{mean}(x) \\
\sigma \leftarrow \mathrm{stdev}(x) \\
Hr \leftarrow \dfrac{x_{15} - x_0}{\sigma} \\
Fr \leftarrow \displaystyle\sum_{i=0}^{15} \left[\left| \left(\mathrm{pnorm}(x_i, m, \sigma) - \frac{i-0.5}{16} \right) \right| \cdot \mathrm{dnorm}(x_i, m, \sigma) \right] \\
MT \leftarrow \displaystyle\sum_{i=0}^{15} \left(\cos\left(\frac{x_i - x_{15}}{1.8} \right) \right) \\
(Hr \;\; Fr \;\; MT \;\; Hr \cdot Fr \;\; Hr \cdot MT \;\; Hr \cdot Fr \cdot MT)^T
\end{array} \right.
$$

Fig. 4. Software for numerical simulation of three examined statistical tests and their multiplicative combinations

Among other things, the above software allows calculating the Hurst-Frozini multiplicative test. Data on the density function of the test's output states are shown in Fig. 5.

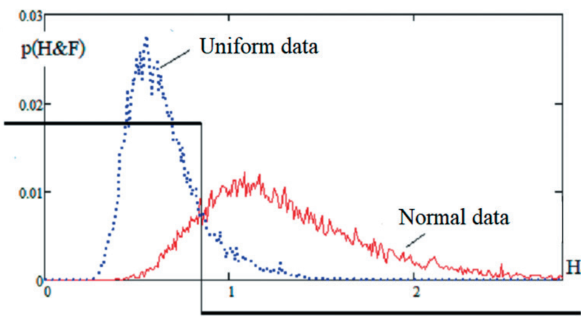

Fig. 5. Density functions of the output states of the Hurst and Frozini multiplicative test

The multiplicative Hurst-Frozini hybrid has a 28% lower probability of errors of the first and second kind $P_1 = P_2 = P_{EE} \approx 0.134$ as compared with the Frozini test, the most powerful one out of them. Another version of the new Hurst (and Murota-Takeuchi) multiplicative statistical test also has a significant, 14% reduction in the probability of errors of the first and second kind as compared with the Murota-Takeuchi test, the most powerful one out of them. The distribution of the response probability functions of the synthesized multiplicative test are shown in Fig. 6.

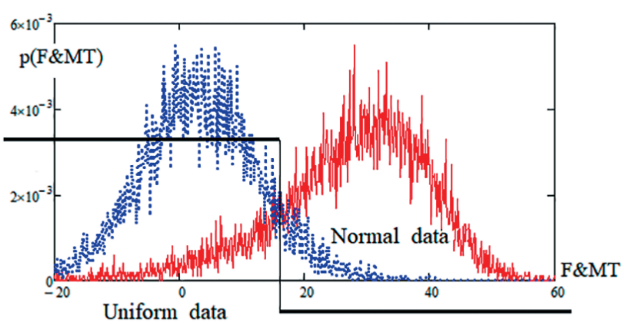

Fig. 6. Density functions of the output states of the Hurst (and Murota-Takeuchi) multiplicative test with probability of error $P_1 = P_2 = P_{EE} \approx 0.133$

Synthesising another test by multiplying the output states of all three examined statistical tests

As the above transformations show, multiplying the responses of two statistical tests significantly reduces the probability of errors of the first and second kind. In theory, the effect should grow if more than two statistical test outputs are multiplied. Fig. 7 shows data on the probability of errors of the first and second kind obtained by multiplying all three examined statistical tests.

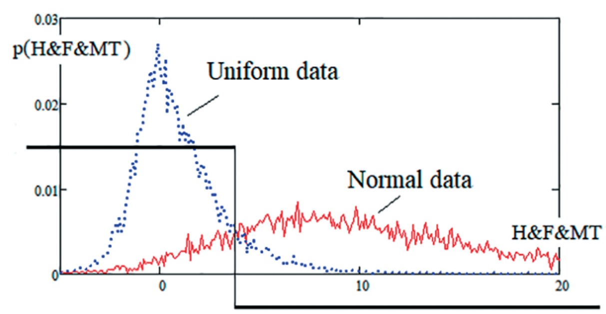

Fig. 7. Density functions of the output states of the Hurst-Frozini-Murota-Takeuchi multiplicative test obtained by multiplying the three examined tests

Multiplying a group of three statistical tests produces the probability of errors of the first and second kind $P_1 = P_2 = P_{EE} \approx 0.136$, which is 1.5% worse as compared with the paired multiplicative tests. It appears that the probability of errors hardly always occurs when the number of multiplied partial parameters grows. Unfortunately, as the number of multiplied tests (the multiplicativity) grows, the correlation of their responses increases as well. That appears to be the exact factor that limits the decrease of the probabilities when it is attempted to increase the number of multiplicatively clustered initial tests.

Conclusion

All known statistical tests can be divided into two classes. In the case under consideration, all three tests belong to the same class. They are similar with respect to the point of equal probability of error in the shared data $P_1 = P_2 = P_{EE}$. The distribution of normal small sample data for the Hurst, Frazini and Murota-Takeuchi tests is always to the left of the point $P_1 = P_2 = P_{EE}$ (see continuous graphs in Fig. 1–3, 5, 6). The distribution of uniform data for such tests is always to the right of the point $P_1 = P_2 = P_{EE}$ (see the dotted graphs in Fig. 1–3, 5, 6). That allows clustering the tests multiplicatively.

On the one hand, multiplicative clustering of known statistical tests allows synthesizing quite a number of new tests. However, such attempts cause a growing rate of correlation of new data, which is a negative phenomenon. In general, synthesizing new statistical tests through multiplicative clustering of existing tests is impossible without taking into account the growing correlations between the new tests.

References

1. GOST R 52633.5-2011. Information protection. Information protection technology. The neural net biometry-code converter automatic training. Moscow: Standartinform; 2018. (in Russ.)

2. Mandelbrot B., Hudson R.L. The (Mis)Behaviour of Markets: A Fractal View of Risk, Ruin and Reward. Moscow, Saint Petersburg, Kiev: Izdatelstvo Viliams; 2006.

3. R 50.1.037-2002. [Applied statistics. Rules for compliance verification of experimental distribution with the theoretical one. Part I. Chi-square-type criteria]. Moscow: Gosstandart of Russia; 2001. (in Russ.)

4. R 50.1.037-2002. [Applied statistics. Rules for compliance verification of experimental distribution with the theoretical one. Part II. Non-parametric tests]. Moscow: Gosstandart of Russia; 2002. (in Russ.)

5. Ivanov A.I., Kupriyanov E.N., Tureev S.V. Neural network integration of classical statistical tests for processing small samples of biometrics data. *Dependability* 2019;2:22-27. DOI: 10.21683/1729-2646-2019-19-2-22-27.

6. Ivanov A.I., Bannykh A.G., Bezyaev A.V. Artificial molecules assembled from artificial neurons that reproduce the work of classical statistical criteria. *Vestnik permskogo unoversiteta. Seria: Matematika. Mekhanika. Informatika* 2020;1(48):26-32. (in Russ.)

7. Kobzar A.I. [Applied mathematical statistics. For engineers and researchers]. Moscow: FIZMATLIT; 2006. (in Russ.)

8. Morelos-Zaragoza R. The R, Art of Error Correcting Coding. Moscow: Tekhnosfera; 2007.

9. Ivanov A.I., Bannykh A.G., Serikova Yu.I. Accounting for the effect of correlations by modulo averaging as part of neural network integration of statistical tests for small samples. *Dependability* 2020; 20(2):28-34. DOI: 10.21683/1729-2646-2020-20-2-28-34.

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The authors’ contribution

Ivanov A.I., Kupriyanov E.N. jointly suggested multiplying outputs of computations based on known statistical tests assuming a steadily increasing quality of the solutions made as the number of multiplied components (tests) grows.

Kupriyanov E.N. conducted a numerical experiment that identified the absence of a steady decrease of the probability of errors of the first and second kind as the number components multiplied as part of synthesis grows.

Conflict of interests

The authors declare the absence of a conflict of interests.