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Efficiency criterion of biased estimates. A new take on old problems

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Abstract. *The perfect case estimation scenario involves unbiased estimation with minimal variance, if such estimate exists. Currently, there are no means of obtaining unbiased estimates (if they do exist!). For instance, a maximum likelihood estimate (NBT test plan) of a mean time to failure T_{mn} = (total operation time)/(number of failures) is highly biased. Those involved in solving applied problems are not satisfied with the situation. Efficient unbiased estimates are used whenever such are available. If it is impossible to find an efficient unbiased estimate in terms of standard deviation, then biased estimate comparison is to be mastered. The vast majority of problems is associated with biased estimates. Within the class of biased estimates, estimates with minimal bias are to be sought, and, among the latter, those with minimal bias. Such estimates in the class of biased estimates should be called bias-efficient or simply efficient, which does not contradict the conventional definition, but only extends it. Such search process guarantees that the obtained estimates are highly accurate. However, with this definition of a bias-efficient estimate, there will always be a pair of compared estimates, in which the total bias of one estimate is slightly higher than that of the other, the same being the case with the total variances of such estimates, but in a different order. In this setting, a formal selection of a bias-efficient estimate becomes impossible and is arbitrary, i.e., the test engineer selects a bias-efficient estimate intuitively. In this case, the test engineer's choice may prove to be incorrect. Thus arises the problem of constructing a criterion of efficiency that would enable a formal selection of a bias-efficient estimate.* **The Aim of the paper.** *The paper aims to build an efficiency criterion, using which the choice of a bias-efficient estimate is unambiguously defined through computation.* **Methods of research.** *To find the bias-efficient estimate, we used integral numerical characteristics of the accuracy of the estimate, namely, the total square of the offset of the expected implementation of a certain variant estimate from the examined parameters of the distribution laws, etc.* **Conclusions.** *1) For the binomial plan and the test plan with recovery and limited test time, performance criteria were constructed that allow unambiguously identifying the bias-efficient estimate out of the submitted estimates. 2) Based on the constructed performance criteria for various test plans, bias-efficient estimates were selected out of the submitted ones.*

Keywords: *estimation, efficient estimation, criterion of efficiency, test plan, biased estimates.*

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Introduction

An efficient estimate is defined as [1]: "An estimate of a parameter that has the lowest expected squared deviation from the estimated parameter for any parameter value is called efficient." The classical theory of mathematical statistics [1] notes that within the class of all possible parameter estimates, there is no efficient estimate. Therefore, the author of [1] further writes: "It is required to impose certain restrictions on the set of estimates, within which we are seeking the best efficient estimate. A natural restriction of the class of estimates is the class of so-called unbiased parameter estimates." In this case, the efficient estimate for the scalar parameter is an unbiased estimate with minimal variance. In some cases, Cramér-Rao inequalities help find the best unbiased estimate [1]: if an estimate is efficient, then, in the above sense, it also is the best, as it has the lowest possible variance.

In estimation, the perfect case scenario involves the use of unbiased estimates with minimal variance, if such estimate exists. For that purpose, in order to identify an efficient estimate, within the class of unbiased estimates, it should be analytically proven that the Cramér-Rao inequality is fulfilled for such estimate. It should be noted that Cramér-Rao inequalities are to be satisfied for all values of the estimated parameters. However, even for exponential families of distributions, for which only efficient estimates exist, an efficient estimation using a Cramér-Rao inequality is only possible for a single function of a parameter. The question is even more relevant as regards families of distributions that are not exponential. If it is difficult to obtain such proof analytically, the total variance should be calculated for all values of the estimated parameter. For an efficient unbiased estimate, the total variance should be minimal.

Currently, there are no means of obtaining unbiased estimates (should such exist!). For instance, a maximum likelihood estimate (NBT test plan) of the mean time to failure T_{mn} = (total operation time)/(number of failures) is highly biased. Those involved in solving applied problems are not satisfied with the situation. Efficient unbiased estimates are used whenever such are available. If it is impossible to find an efficient unbiased estimate in terms of mean square variance, then biased estimate comparison is to be mastered. The vast majority of problems is associated with biased estimates.

Within the class of biased estimates, estimates with minimal bias are to be sought, and, among the latter, those with minimal variance [2]. Such estimates in the class of biased estimates should be called bias-efficient or simply efficient, which does not contradict the conventional definition, but only extends it. Such search process guarantees that the obtained estimates are highly accurate. Note that the experience of constructing efficient estimates shows that the resulting unbiased efficient estimate will not always have a minimum variance [2]. Rather, on the contrary, there will always be an estimate that has minimal variance compared to the unbiased estimate. In all cases where there is an efficient (unbiased) estimate, there is a biased estimate that is more accurate than the efficient one, i.e., with a smaller squared error [3, p. 284]. That fact favours bias as the primary factor in constructing the evaluation efficiency criterion. In order to determine the bias-efficient estimate, the total biases and variances are to be calculated for all values of the estimated parameter. For an efficient biased estimate, each sum must be minimal. Such definition of an efficient estimate within a particular class of biased estimates does not contradict the definition of an efficient estimate within a class of unbiased estimates. On the contrary, defining an efficient estimate within a class of unbiased estimates is a frequent case of defining an efficient estimate within a certain individual class of biased estimates that includes a subclass of unbiased estimates.

Why the integral approach? When comparing using the classical method, whereas the variance should be minimal for all parameter values at once, we deduce that one of the compared biased estimates will have a lower variance in one part of the parameter values, while the other will have a lower variance in the remaining part, with a comparable bias. Comparing them is what the summation of all variances (biases) is required for. The sums of biases and variances define the efficiency criterion.

However, with this definition of a bias-efficient estimate, there will always be a pair of compared estimates, in which the total bias of one estimate is slightly higher than that of the other, the same being the case with the total variances of such estimates, but in a different order. In this setting, a formal selection of a bias-efficient estimate becomes impossible and is arbitrary, i.e., the test engineer selects a bias-efficient estimate intuitively. In this case, the test engineer's choice may prove to be incorrect. Thus arises the problem of constructing an efficiency criterion that would enable a formal selection of a bias-efficient estimate.

The Aim of the paper

The paper aims to build an efficiency criterion, using which the choice of a bias-efficient estimate is unambiguously defined through computation.

Methods of research

The bias-efficient estimate was found using integral numerical characteristics of the accuracy of estimate, i.e., the sum square of the bias of the expected realization of an estimate from the considered parameters of the distribution laws, etc. [2].

Constructing the estimate efficiency criterion

Let us denote by $A(\theta)$ the total bias of estimate θ from estimated parameter t, and by $B(\theta)$ the total variance of estimate θ from estimated parameter t. Note that summation is done within the operating range both for all values of estimated parameter t, and all values of the test plan and other parameters (e.g., time it takes to estimate the probability of no failure (PNF).

For the purpose of constructing an efficiency criterion of biased estimates we will characterize arbitrary statistical estimate θ by bias and variance. Let us denote by $b = E(\theta) - t$ the bias of estimate θ from parameter t, where E is the mathematical expectation, and by D the variance of estimate θ. Then the variance (in the mean square sense) of a certain estimate θ from the estimated parameter t is expressed by the following formula [1, 4, 5]:

$$B(\theta) = E(\theta - t)^2 = D + b^2. \quad (1)$$

Note that, when dispersion changes, the variance as an efficiency characteristic also changes by the same value (see formula (1)). That is, it changes regardless of the dependence on the specific value of estimate bias. Let us try to associate the dispersion and bias square in such a way as to make the variance change adjusted to bias whenever dispersion variates. We will take into consideration the fact that bias is the primary factor in choosing an efficient estimate. The newly built characteristic $C(\theta)$ must be such as, when the dispersion changes by the value of δD, for small biases $b \approx 0 + \delta$, the adjustment for the effect of the bias on the characteristic was insignificant, and vice versa, for large biases $b >> 0$, the adjustment for the effect of bias on characteristic $C(\theta)$ was significant. We will require that the variation of characteristic $C(\theta)$ was linear with respect to characteristics D and b^2. The product of characteristics D and b^2 fulfils this requirement to the fullest:

$$C(\theta) = D \cdot b^2. \quad (2)$$

Out of formula (2) follows that, as dispersion changes by value δD, characteristic $C(\theta) = (D + \delta D)\cdot b^2 = D\cdot b^2 + \delta D\cdot b^2$ changes by a value that takes into account the squared bias linearly. The opposite is also true, i.e., when the squared bias changes by a certain value, characteristic $C(\theta)$ changes by a value that takes into account the dispersion value linearly. Figuratively speaking, characteristic $C(\theta)$ reflects on the Cartesian axes D and b^2 as a rectangle with the area of $D\cdot b^2$. Any slight change to characteristics D and b^2 modifies the area or configuration of the rectangle. Thus, in case of slightly different characteristics D and b^2, the estimate with the minimum characteristic $C(\theta)$ (area) should be chosen as the bias-efficient. If characteristics $C(\theta)$ (areas) are equal, the estimate with the lowest bias should be chosen as the bias-efficient. Let it be reminded that the criterion was constructed only for biased estimates. In the case of unbiased estimates, variance $B(\theta)$ (see formula (1)) is such characteristic (criterion). Note that, for unbiased estimates, their realizations are grouped around the true quantitative value of the estimated parameter from different sides. When defining the efficiency criterion, similar properties are to be required from biased estimates.

Let us define the requirements for the process of selecting bias-efficient estimates:

– the proposed estimates must be strictly monotonous in all their parameters;

– estimates with a minimum bias of $A(\theta) = b^2$ or close to such are selected.

If, in the process of selection out of a number submitted estimates, there is a single unbiased estimate, then the latter is the bias-efficient one. For this estimate to be efficient in the class of unbiased estimates, it is required to prove the Cramér-Rao for such estimate:

– estimates, for which inequality $A = b^2 > D$ is fulfilled, i.e., the bias prevails over the value scatter of such estimate, are excluded;

– estimates are selected, for which the inequality $D/A > 4$ is fulfilled, i.e., the estimates, for which the realizations are grouped around the true quantitative value of the estimated parameter from different sides;

– out of the remaining estimates, the estimate with the minimum bias $A(\theta) = b^2$ or close to such (+5 ... +20%) is selected. In the case a single estimate with minimum bias A was selected, such estimate is considered bias-efficient;

– in case A are equal, the estimate with minimal variance is chosen as the bias-efficient one.

The majority of manipulations is replaced by the proposed criterion $C(\theta) = D\cdot b^2$.

Let us consider examples of constructing a criterion for bias-efficient estimate selection.

Binomial test plan. Probability of no failure

Here and further, we will use the findings of [2]. Let us denote by θ a certain abstract estimate of the probability of failure in the course of testing of n products. We will limit the scope of testing to $0 < n \le 10$, which is the cost limit for highly dependable and complex products. Then, the total bias formula will be as follows:

$$A\big(\theta(n;R)\big) = \frac{1}{10}\sum_{n=1}^{10}\int_0^1 \big(E\theta(n;R) - p\big)^2\, dp.$$

The formula for the total variance is as follows:

$$D\big(\theta(n;R)\big) = \frac{1}{10}\sum_{n=1}^{10}\int_0^1 E\big(\theta(n;R) - E\theta(n;R)\big)^2\, dp.$$

Let us note that the probability function of the binomial test plan P_Σ steadily decreases as p grows [5], therefore, equations $P_\Sigma(R = r) = \sum_{k=0}^{r} P_n(k, w) = 0{,}5 + x$ and $P_\Sigma(R = r) = \sum_{k=0}^{r} P_n(k, v) = 0{,}5$ have a single solution, where $P_n(k,p) = C_n^r p^r (1-p)^{n-r}$.

Calculations show that probability $\gamma = 0.5 + x = 0.8181$ corresponds to estimate w that minimizes functional $A(\theta(n;R))$. Table 1 shows the results of substituting into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ of the following failure

Table 1. Results of substituting the proposed failure probability estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the binomial test plan

Type of functional	v $\gamma = 0.5$	w $\gamma = 0.81$	p_1 $\gamma = 0.5$	p_2 $\gamma = 0.81$	p_3 $\gamma = 0.81$	$u = (R + 1)/(n + 2)$	$p_0 = R / n$
A	0.0176	0.0037	0.0113	0.0015	0.0070	0.0104	$6 \cdot 10^{-33}$
D	0.0270	0.0402	0.0288	0.0401	0.0226	0.0162	0.0488
D / A	1.53	10.86	2.54	26.73	3.22	1.55	∞
$C = D \cdot A \cdot 10^4$	4.752	1.4874	3.2544	0.6015	1.595	1.6848	10^{-30}

Table 2. Results of substitution of failure probability estimates ^v, ^w, p_{10}, p_{20} into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the binomial test plan

Type of functional	^v $\gamma = 0.5$	^w $\gamma = 0.81$	p_{10} $\gamma = 0.5$	p_{20} $\gamma = 0.81$
A	0.0034	0.0030	0.000680	0.000355
D	0.0356	0.0427	0.0425	0.0443
D / A	10.47	14.23	62.5	124.7
$C = D \cdot A \cdot 10^4$	1.210	1.28	0.289	0.157

probability estimates: v, w, $p_0 = R / n$, p_1, p_2, p_3 [5] and $u = (R + 1)/(n + 2)$, where

$p_1 = v(0.5;n)$, $R = 0$ and $p_1 = R / n$, $R > 0$;

$p_2 = w(0.81;n)$, $R = 0$ and $p_2 = R / n$, $R > 0$;

$p_3 = w(0.81;n)$, $R = 0$ and $p_3 = u$, $R > 0$.

Functionals $A(\theta(n;R))$ and $D(\theta(n;R))$ were calculated with the step of $\partial p = 10^{-3}$. Implicit estimates w and v were calculated with the accuracy of 10^{-4}.

Here and further, for the purpose of table construction, as part of calculation of characteristic $C = D \cdot A$, functionals A and D were calculated for each value of parameters n and p with subsequent individual summation, and based on the obtained total values of A and D, characteristic $C = D \cdot A$ was calculated.

Note that calculating characteristic C directly as a functional

$$C\left(\theta\left(n;R\right)\right) = \frac{1}{10}\sum_{n=1}^{10}\int_0^1 E\left\{\theta\left(n;R\right) - E\theta\left(n;R\right)\right\}^2 \cdot \left\{E\theta\left(n;R\right) - p\right\}^2 dp$$

is associated with great computational difficulties due to the limited word length in the computer system, which, in the course of computation, causes clearing of significant summable values. That affects the final result.

Unbiased estimate $p_0 = R / n$ that was given for comparison is excluded from consideration as a bias-efficient one despite the fact that it is efficient.

Out of Table 1 follows that estimates v, p_1, p_3, u are to be excluded from consideration, as inequality $D / A > 4$ does not apply to them. Then, out of Table 1 also follows that estimates w and p_2 have minimal and comparable biases. Their values do not differ by more than $(0.0037 - 0.0015) \cdot 100/0.0037 = 59\%$. In accordance with the proposed efficiency criterion of biased estimates, estimate p_2 is to be definitely considered efficient. Out of the construction follows that the criterion constructed based on characteristic $C = D \cdot A$ unambiguously determines the bias-efficient estimate without recurring to most of the above reasonings in this paragraph.

The proposed estimates v, w, p_1, p_2 for the binomial test plan have a bias that can be reduced, which slightly modifies the estimates as follows:

^$v = v(0.5;n,R) - 0.4 / ((R + 1)n)$;

^$w = w(0.81;n,R) - 0.1/((R+1)n)$;

$p_{10} =$ ^$v(0.5;n)$, $R = 0$ and $p_{10} = R / n$, $R > 0$;

$p_{20} =$ ^$w(0.81;n)$, $R = 0$ and $p_{20} = R / n$, $R > 0$.

Table 2 shows the results of substituting into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ of the following probability of failure estimates: ^v, ^w, p_{10}, p_{20}.

Out of Table 2 follows that, for all available estimates, inequality $D / A > 4$ is correct. In accordance with the proposed efficiency criterion of biased estimates, estimate p_{20} is to be definitely considered efficient.

Binomial test plan. Mean time to failure

Let us assume that the products' time to failure follows the exponential law of probability distribution (d.l.) with parameter T_0, where the latter is identical to the mean time to failure. Let us denote the test time of each of the N products as τ.

As the criterion of efficient MTF estimate, a functional is constructed that is based on summing the squared relative biases of expected estimates $\theta(R,n)$ from the parameter t of the exponential d.l. (MTF) for all possible values of N, τ, $T_0 = t$ [2]

$$A\left(\theta\left(n;R\right)\right) = \frac{1}{3}\sum_{\tau=10^3}^{10^5}\frac{1}{10}\sum_{n=1}^{10}\int_0^\infty \frac{1}{t^2}\left\{E\theta\left(n;R,\tau\right) - t\right\}^2 dt.$$

Integration is done for all possible values of parameter (MTF) t out of $[0;\infty]$.

The formula for total variance D is

$$D\left(\theta\left(n;R\right)\right) = \frac{1}{3}\sum_{\tau=10^3}^{10^5}\frac{1}{10}\sum_{n=1}^{10}\int_0^\infty \frac{1}{t^2} E\left\{\theta\left(n;R,\tau\right) - E\theta\left(n;R,\tau\right)\right\}^2 dt.$$

Table 3. Results of substitution of suggested MTF estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the binomial test plan

Type of functional	T_1	T_2	T_3	T_4	T_5	T_6
A	1513	11.27	11.26	11.09	11.01	10.59
D	1.962	3.679	7.402	7.534	4.983	9.157
D / A	≈0.01	0.32	0.65	0.67	0.45	0.86
$C = D \cdot A$	2968	41.4	83.3	83.6	54.8	96.9

Table 4. Results of substitution of suggested MTF estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the binomial test plan

Type of functional	T_{10}	T_{20}	T_{30}	T_{40}	T_{50}	T_{60}
A	5.67	4.62	5.34	5.27	5.03	4.85
D	9.65	7.06	3.62	3.69	4.98	5.47
D / A	1.70	1.52	0.67	0.70	0.99	1.12
$C = D \cdot A$	54	32.61	19.33	19.44	25.04	26.52

Table 3 shows the results of substitution of the following MTF estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$:

$T_1 = ((n - R)\cdot\tau + R\cdot\tau / 2) / (R + 1)$;

$T_2 = -\tau / \mathrm{Ln}(1 - (R + 1) / (n + 1))$;

$T_3 = -\tau / \mathrm{Ln}(1 - p_1)$;

$T_4 = -\tau/\mathrm{Ln}(1 - p_4)$, where $p_4 = u = (R + 1) / (n + 2)$, $R = 0$ and $p_4 = p_0 = R / n$, $R > 0$;

$T_5 = -\tau / \mathrm{Ln}(1 - v(R,n,\gamma = 0.5)$;

$T_6 = -\tau/\mathrm{Ln}(1 - v(R,n,\gamma = 0.62)$.

Out of Table 3 follows that, in accordance with the constructed criterion, all estimates are to be excluded from consideration, as the critical condition $D / A > 4$ is not fulfilled for them. However, due to the need to make a choice, estimate $T_6 = -\tau/\mathrm{Ln}(1 - v(R,n,\gamma = 0.62))$ with a minimum bias and maximum characteristic $D / A = 0.86$ should be considered conditionally bias-efficient.

The proposed MTF estimates for the binomial test plan are strongly biased, yet this bias can be reduced. The type of estimates will change slightly as follows:

$T_{10} = 400 + 0.015\cdot\tau + \tau\cdot(n - R + R\cdot 0.02) / (R + 0.5))$;

$T_{20} = 400 + 0.015\cdot\tau + (-\tau\cdot 0.7/\mathrm{Ln}(1 - (R + 0.4)/(n + 0.4)))$;

$T_{30} = 400 + 0.015\cdot\tau + (-\tau\cdot 0.7 - \tau/\mathrm{Ln}(1 - p_1))$;

$T_{40} = 400 + 0.015\cdot\tau + (-\tau\cdot 0.7/\mathrm{Ln}(1 - p_4))$, where $p_4 = u=(R + 1)/(n + 2)$, $R = 0$ and $p_4 = p_0 = R / n$, $R > 0$;

$T_{50} = 400 + 0.015\cdot\tau + (-\tau/\mathrm{Ln}(1 - v(R,n,\gamma = 0.5))$;

$T_{60} = 400 + 0.015\cdot\tau + (-\tau\cdot 0.75/\mathrm{Ln}(1 - v(R,n,\gamma = 0.62))$.

Variants of the suggested estimates with smaller biases are shown in Table 4.

Out of Table 4 follows that, in accordance with the constructed criterion, all estimates are to be excluded from consideration, as critical condition $D / A < 4$ is fulfilled. However, as a choice has to be made, the minimum bias estimate $T_{20} = 400 + 0.015\cdot\tau - \tau\cdot 0.7 / \mathrm{Ln}(1 - (R + 0.4) / (n + 0.4))$ should be regarded as conditionally bias-efficient.

Further reducing the bias on the selected class of estimates would be quite challenging. In this case, the problem of bias reduction is solved by searching a wider class of estimates that includes a class of unbiased or similar estimates. Note that the closer an estimate is to unbiased (characteristic A tends to zero), if it exists, its variance increases (see Table 1), below tending to the variance of an unbiased estimate, or decreases, above tending to the variance of an unbiased estimate, which forces their realizations to cluster around the true quantitative value of the estimated parameter from different sides similarly to the realizations of unbiased estimates. This fact follows directly from the Cramér-Rao inequality for biased estimates [5, f. 2.14.14]. Therefore, for estimates with a near-zero bias, condition $D / A > 4$ will always be fulfilled. It is important to note that the estimates of the selected class intended for finding bias-efficient estimates are to be strictly monotone with respect to all parameters (R, τ, n).

NB test plan. MTF

In what follows, the designations of the test plan are according to [6, 7]. For the $NB\tau$ plan, the number of observed failures (r) is a sufficient statistic [6, 7]. Let us denote a random number of failures as R, then, for a $NB\tau$ test plan, the random value R (hereinafter referred to as r.v.) has a Poisson distribution $L(r;\Delta)$ with the parameter $\Delta = n\tau / T_0$, $n = N$ [4–7]. Then, by definition, r is the realization of r.v. R. On the other hand, R is the sum of r.v. X_i, each of which is a random number of failures of one of the N tested products ($1 < i < n$). R.v. X_i have a Poisson distribution with parameter Δ / n

$$L(r;\Delta) = \sum_{k=0}^{X_1+\ldots+X_n=r} \exp\{-\Delta\}\cdot\frac{\Delta^k}{k!}. \quad (3)$$

Let us use formula (3) and examine the properties of the parameter estimate Δ obtained from the equation

$$L(r;\Delta) = \sum_{k=0}^{r} \exp\{-\Delta\}\cdot\frac{\Delta^k}{k!} = 0{,}5 \text{ or}$$

$$\varepsilon(\Delta) = \ln(2) + \ln\left(\sum_{k=0}^{r}\frac{\Delta^k}{k!}\right) - \Delta. \quad (4)$$

Minimizing the absolute value $\varepsilon(\Delta)$ in formula (4), with the required accuracy, we obtain the sought point estimate of

Table 5. Results of substituting the suggested PNF estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the $NB\tau$ test plan.

Type of functional	A	D	D/A	$C=D\cdot A$
$T_{11} = 2.2n\tau$ if $R = 0$ and $T_{11} = n\tau / (R + 1 + 1 / R)$ if $R > 0$	0.214	3.93	18.36	0.841
$T_{10} = 2.1n\tau$ if $R = 0$ and $T_{10} = n\tau / (R + 1.2)$ if $R > 0$	0.234	3.89	16.62	0.910
$T_6 = 1.5n\tau / \Lambda$ if $R = 0$ and $T_6 = n\tau / (\Lambda + 0.5)$ if $R > 0$	0.234	3.98	17.00	0.931
$T_1 = 2n\tau$ if $R = 0$ and $T_1 = n\tau / (R + 1)$ if $R > 0$	0.25	4.12	16.48	1.03
$T_8 = n\tau / (R + 1) + n\tau 10^{-(R+0.5)} / (R + 0.5)$	0.28	4.00	14.28	1.134
$T_7 = n\tau / (R + 1) + n\tau e^{-(R+1)} / (R + 1)$ [8]	0.34	4.1	12.05	1.394
$T_9 = n\tau / (R+0.7)$	0.364	4.43	12.17	1.61
$T_5 = n\tau / \Lambda$	0.37	4.51	12.18	1.66
$T_3 = n\tau / (R + 1)$	0.500	3.72	7.44	2.30
$T_2 = 2n\tau$ if $R = 0$ and $T_2 = n\tau / R$ if $R > 0$	1.437	7.94	5.52	11.40
$T_4 = 6n\tau$ if $R = 0$ and $T_4 = n\tau / (R + 0.5)$ if $R > 0$	5.36	10.21	1.90	54.72

Table 6. Results of substituting the proposed PNF estimates into functionals $A(\theta(m,g;R))$, $D(\theta(m,g;R))$ for the $NB\tau$ test plan

Type of functional	e^{-g/T_1}	e^{-g/T_2}	e^{-g/T_3}	e^{-g/T_4}	e^{-g/T_5}	e^{-g/T_9}	e^{-g/T_7}
A	0.0346	0.0300	0.0641	0.0156	0.0410	0.0157	0.0458
D	0.0987	0.1066	0.0740	0.1501	0.0876	0.1486	0.0851
D / A	2.85	3.55	1.15	9.62	2.13	9.46	1.85
$C = D\cdot A\cdot 10^3$	3.415	3.198	47.43	2.341	35.91	2.333	3.914

the Poisson parameter $\Lambda = \Lambda(R)$. Having estimate $\Lambda(R)$, we easily obtain the MTF estimate $T_5 = n\tau / \Lambda$. Let us examine the following MTF estimates:

– implicit estimate $T_5 = n\tau / \Lambda$;
– $T_1 = 2n\tau$ if $R = 0$ and $T_1 = n\tau / (R + 1)$ if $R > 0$;
– $T_2 = 2n\tau$ if $R = 0$ and $T_2 = n\tau / R$ if $R > 0$;
– $T_3 = n\tau / (R + 1)$;
– $T_4 = 6n\tau$ if $R = 0$ and $T_4 = n\tau / (R + 0.5)$ if $R > 0$;
– $T_6 = 1.5n\tau / \Lambda$ if $R = 0$ and $T_6 = n\tau / (\Lambda + 0.5)$ if $R > 0$;
– $T_7 = n\tau / (R + 1) + n\tau e^{-(R+1)} / (R + 1)$ [8];
– $T_8 = n\tau / (R + 1) + n\tau 10^{-(R+0.5)} / (R + 0.5)$;
– $T_9 = n\tau / (R + \beta(R))$ if $\beta = 0.7$;
– $T_{10} = 2.1n\tau$ if $R = 0$ and $T_{10} = n\tau / (R + 1.2)$ if $R > 0$;
– $T_{11} = 2.2n\tau$ if $R = 0$ and $T_{11} = n\tau / (R + 1 + 1 / R)$ if $R > 0$.

These bias estimates are based on functional ($T_0 = t$) [2]

$$A\left(\theta\left(n;R\right)\right)=\int_0^\infty \frac{1}{t^2}\left\{E\theta\left(n;R\right)-t\right\}^2 d\Delta.$$

The formula for the normalized variance D is

$$D\left(\theta\left(n;R\right)\right)=\int_0^\infty \frac{1}{t^2}E\left\{\theta\left(n;R\right)-E\theta\left(n;R\right)\right\}^2 d\Delta.$$

Table 5 shows the results of substituting the suggested PNF estimates into functionals $A(\theta(n;R))$, $D(\theta(n;R))$ for the $NB\tau$ test plan.

Out of Table 5 follows that estimates T_1, T_6, T_8, T_{10} and T have approximately the same biases. The greatest difference between their values is $(0.28 – 0.214)\cdot 100 / 0.28 = 23$ %. In accordance with the suggested efficiency criterion of biased estimates, estimate T_{11} with the minimum value of characteristic $C = 0.841$ must certainly be regarded as the most efficient.

Note that [2] provides the evidence of the fact that, in the class of estimates $T_R = n\tau / (R + 1) + n\tau f(R)$, estimate $T_1 = 2n\tau$ if $R = 0$ and $T_1 = n\tau / (R + 1)$ if $R > 0$ affords a minimum to functional $A = 0.25$. Let us prove that estimate $T_9 = n\tau / (R + \beta(R))$ does not belong to the class of estimates T_R, for which it suffices to represent estimate T_9 as $T_9 = n\tau(R + 2) / (R + 1)(R + \beta(R)) – n\tau / (R + 1)(R + \beta(R))$, hence the statement. The only estimate out of class T_9 that belongs to the class of estimates T_R is the estimate of type

$T_9 = n\tau / (R + \beta(R)) = n\tau(R + 2) / (R + 1)(R + \beta(R)) – n\tau / (R + 1)(R + \beta(R)) = = n\tau(R + 2) / (R + 1)(R + 2) – n\tau / (R + 1)(R + 2) = n\tau / (R + 1) – n\tau / (R + 1)(R + 2)$

if $\beta(R) = 2$ (or if $\beta(R) = 0$, i.e., $T_2 = 2n\tau$ if $R = 0$ and $T_2 = n\tau / R = n\tau / (R + 1) + n\tau / R(R + 1)$ if $R > 0$). Where it is easy to see that $n\tau f(R) = –n\tau / (R + 1)(R + 2)$. Therefore, the occurrence of the values of the functional $A(T_{10}) = 0.234 < 0.25$ on estimate T_{10} and $A(T_{11}) = 0.214 < 0.25$ on estimate T_{11} is quite justified.

NB test plan. Probability of no failure

Let us denote $m = n\tau$. Let us examine the PNF estimates for the time interval g of the form $\theta(m,g;R) = \exp\{-g / T_i\}$, where T_i is a certain MTF estimate (see Table 5). Instead of estimate T_6, let us examine estimate $T_9 = 4n\tau / \Lambda$ if $R = 0$ and $T_9 = n\tau / \Lambda$ if $R > 0$.

The comparison the PNF estimates in terms of the total bias value is based on a functional of the form [2]

$$A\left(\theta\right)=\frac{1}{3}\sum_{m=10^3}^{10^5}\frac{1}{10}\sum_{g=10^3}^{10^5}\int_0^\infty \frac{1}{t^2}\left\{E\theta\left(n;R,m,g\right)-\exp\left(-g\Delta/m\right)\right\}^2 d\Delta.$$

The formula for the normalized variance D is

Table 7. Results of calculating the PNF of Example 1 ($\tau = g$, R = 0)

$N = n$	$p_{20} = \hat{}w(0.81;n)$, $R = 0$ and $p_{20} = R / n$, $R > 0$; $P_{20} = 1 - p_{20}(R = 0) = 1 - \hat{}w(\gamma = 0.81, R = 0)$ Binomial plan	$P_{NB\tau}(T_9) = \exp\{-g\Lambda / 4n\tau\}$, $g = \tau$, $R = 0$, $\Lambda(R) = 0.693148$ $NB\tau$ plan
1	0.91	0.841
2	0.95	0.917
3	0.965	0.944
4	0.973	0.958
5	0.978	0.966
6	0.982	0.972
7	0.984	0.976
8	0.986	0.979
9	0.988	0.981
10	0.989	0.983

Table 8. Results of MTF calculation for Example 2 (τ = 1000, R = 0)

$N = n$	$T_{20} = 400 + 0.015\cdot\tau + (-\tau\cdot 0.7 / \mathrm{Ln}(1 - (R + 0.4)/(n + 0.4)))$ Binomial plan	$T_{11} = 2.2n\tau$, if $R = 0$ and $T_{11} = n\tau / (R + 1 + 1 / R)$, if $R > 0$ $NB\tau$ plan
1	2495	2200
2	4254	4400
3	6008	6600
4	7759	8800
5	9511	11000
6	11261	13200
7	13012	15400
8	14762	17600
9	16512	19800
10	18263	22000

$$D(\theta) = \frac{1}{3}\sum_{m=10^3}^{10^5}\frac{1}{10}\sum_{g=10^3}^{10^5}\int_0^\infty \frac{1}{t^2} E\left\{\begin{matrix}\theta(n;R,m,g)- \\ -E\theta(n;R,m,g)\end{matrix}\right\}^2 d\Delta.$$

Table 6 shows the results of substituting the proposed PNF estimates into functionals A ($\theta(m,g;R)$), $D(\theta(m,g;R))$ for the $NB\tau$ test plan.

Out of Table 6 follows that estimates e^{-g/T_4} and e^{-g/T_9} have approximately the same biases. Their values differ by (0.0157 – 0.0156) · 100/0.0157 = 0.63%. According to the proposed efficiency criterion of biased estimates, estimate e^{-g/T_9} with the minimum value of characteristic C = 2.333 is to be regarded as the most efficient.

Example 1. In the course of dependability testing of a set of 1, 2, ..., 10 products, no failures occurred. It is required to estimate the PNF of the inspected batch of products using bias-efficient estimates for the binomial test plan and the test plan with recovery and limited test time. The calculation results are given in Table 7.

Out of Example 1 follows that for the binomial plan and the test plan with recovery and limited test time, in the setting of Example 1, the bias-efficient estimates differ (in case of $R = 0$). It is up to the test engineer to choose which estimates to use in this case.

Example 2. In the course of 1000-hour dependability tests of a set of 1, 2,..., 10 products, no failures occurred. It is required to estimate the MTF of the inspected batch of products using efficient estimates for the binomial test plan and the test plan with recovery and limited test time. The calculation results are given in Table 8.

Out of examples 1 and 2 follows that for the binomial plan and the test plan with recovery and limited test time, the outputs of bias-efficient estimates differ (case of $R = 0$). It is up to the test engineer to choose which estimates to use in this case.

Afterword

A general approach is defined to constructing an efficiency criterion of biased estimates. For various test plans, performance criteria were constructed that allow unambiguously identifying the bias-efficient estimate out of those submitted. However, the problem of constructing (obtaining) efficient estimates (biased and not) with good statistical properties remains at the focus of the dependability theory and awaiting a solution.

Conclusions

1) For the binomial plan and the test plan with recovery and limited test time, performance criteria were constructed that allow unambiguously identifying the bias-efficient estimate out of the submitted estimates.

2) Based on the constructed performance criteria for various test plans, bias-efficient estimates were selected out of the submitted ones.

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The author's contribution

The author proposed a new criterion of bias-specific efficiency and used it to obtain bias-efficient estimates of various test plans.

Conflict of interests

The author declares the absence of a conflict of interests.