Use of exponential distribution in mathematical models of dependability

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Abstract. *The exponential distribution of time to event or end of state is popular in the dependability theory. This distribution is characterized by the strength that is a convenient parameter used in mathematical models and calculations. The exponential distribution is used as part of dependability-related process simulation. Examples are given to illustrate the applicability of the exponential distribution.* **Aim.** *The aim of the paper is to improve the dependability-related simulation methods when using the exponential distribution of periods of states or times to events.* **Methods.** *The assumption of the exponential distribution of time between events can be justified or discarded using methods of the probability theory and/or mathematical statistics or on the basis of personal or engineering experience. It has been experimentally established that the failure flow in an established mode of operation is stationary, ordinary and produces no consequences. Such flow is Poisson and is distinct in the fact that the time between two consecutive failures is distributed exponentially with a constant rate. This exponential distribution is reasonably extended to the distribution of an item's failure-free time. However, in other cases, the use of exponential distribution is often not duly substantiated. The methodological approach and the respective conclusions are case-based. A number of experience-based cases are given to show the non-applicability of exponential distribution.* **Discussion**. *Cases are examined, in which the judgement on the applicability or non-applicability of exponential distribution can be made on the basis of personal experience or the probability theory. However, in case of such events as completion of recovery, duration of scheduled inspection, duration of maintenance, etc., a judgement regarding the applicability of exponential distribution cannot be made in the absence of personal experience associated with such events. The distribution of such durations is to be established using statistical methods. The paper refers to the author's publications that compare the frequency of equipment inspections with regular and exponentially distributed periods. The calculated values of some indicators are retained, while for some others they are different. There is a two-fold difference between the unavailability values for the above ways of defining the inspection frequency.* **Findings and conclusions.** *The proposed improvements to the application of exponential distribution as part of dependability simulation come down to the requirement of clear substantiation of the application of exponential distribution of time between events using methods of the probability theory and mathematical statistics. An unknown random distribution cannot be replaced with an exponential distribution without a valid substantiation. Replacing a random time in a subset of states with a random exponentially distributed time with a constant rate should be done with an error calculation.*

Keywords: *dependability, exponential distribution, event rate.*

Introduction

Exponential distribution is widely used in mathematical dependability models. The advantage of this distribution is that it is characterized by a single parameter, i.e., the event rate, which gives the model simplicity. In particular, a model with a constant event rate allows using Markovian methods. Event rates are also used while generating and solving differential equilibrium equations as part of transitions between states, which allows obtaining state probabilities in both a transient and steady state.

The exponential distribution of time to failure is substantiated using probabilistic and statistical methods. It has been experimentally established that an item's failure is a random event, while the failure flow in an established mode of operation is stationary, ordinary and produces no consequences. Such flow is of Poisson type; it has a simple analytical description. The characteristic feature of a Poisson flow is that the time between two consecutive failures is distributed exponentially with a constant rate. This exponential distribution is reasonably extended to the distribution of an item's failure-free time.

The exponential distribution simulates the random time between two consecutive events. The exponential distribution is also extended to various states and events. It is used for the duration of equipment recovery (repair), time between inspections of the technical state of equipment and for other cases. On the Internet, there are many use cases of exponential distribution and of associated problems.

However, in a number of cases, the use of exponential distribution is often not duly substantiated. The following rationale is presented:

– an exponential distribution of any random time period is used similarly to the distribution of the time to failure;

– the time period is random, so it is exponentially distributed;

– exponential distribution is conveniently used in mathematical models;

– everyone uses exponential distributions, so do I;

– in literature, there are many mentions of constant or random time periods with an unknown distribution being replaced with an exponential distribution;

– exponential distribution is commonly used;

– the transition from a constant-time state to a random-time state is due to the requirement of simulation.

Such substantiations are what might be called a sham. Hence, if exponential distribution was adopted without due substantiation, its use within mathematical models may be erroneous or unacceptable.

Let us try working out a substantiation for using exponential distribution.

Source overview

The failure rate as a parameter of exponential distribution is featured in many state standards: [2, 7, 8, 9, 10]. The restoration rate is referred to in [2, 9, 10], while the repair rate is mentioned in [7, 8]. Random maintenance (repair) duration is used in [5].

[10] describes the advantages of using the Markovian methods for the purpose of dependability research of various systems, as well as assumptions and limitations for cases where the failure and restoration rates are constant in time. The assumption of constant restoration rate is to be substantiated if the mean restoration time is not negligible compared to the corresponding mean time to failure. [10] also states that the state transition rates are used not only for failures and restorations. Such transitions may be caused by a variety of events.

According to [17], the assumption of exponential distribution is not always justified. That is especially true for the restoration time, as the assumption that the remaining restoration time is independent from the already spent time appears to be quite unnatural. However, if the average time to failure is significantly longer than the restoration time, many dependability indicators do not depend on the type of restoration time distribution.

The use of exponential distribution in dependability is widely covered in scientific and training literature, e.g., [15]. It should be noted that, in the dependability theory, not only the exponential, but other distributions are used, if required: normal, Weibull, binomial, Poisson, gamma [14, 16].

Statistical methods are also widely described in literature. A number of state standards are dedicated to such methods. Thus, [3] lists procedures intended for item reliability indicator calculation based on data on similar items, operation and testing. Standard [6] establishes statistical methods for calculating point estimates, confidence, prediction and tolerance intervals for failure rates of items whose times to failure are exponentially distributed. The above quantitative methods are applicable to the rates of other events, times to which are exponentially distributed.

Standard [4] is intended for ensuring the safety, availability and cost-effective operation of items. Failure management involves maintenance, modification of application rules and other actions aimed at mitigating the impact of failures. The standard provides guidance on planning and performing reliability tests and applying statistical methods for analysing test data.

Method. Use cases of exponential distribution

Thus, the use of constant rate of various events (states) in Markovian models requires serious substantiation. The assumption of exponential distribution of time between events can be justified or discarded in several ways, e.g.:

1) using the methods of mathematical statistics;

2) using the methods of the probability theory;

3) on the basis of personal or engineering experience.

State standards describe the application of methods of mathematical statistics in sufficient detail.

A judgement regarding the applicability or non-applicability of exponential distribution may be made based on the

assumption that the remainder of time is independent from the already spent time [17]. Additionally, a judgement on the applicability or non-applicability of exponential distribution may be made based on the personal experience of a modern person. The above use cases of exponential distribution are based on the meaning and personal experience.

First, let us set forth a value table for functions $P(t) = exp(-\lambda t)$ and $F(t) = 1 - exp(-\lambda t)$ in a number of points. This will allow analysing the above cases with no additional calculations.

Table 1. Values of functions $P(t)$ and $F(t)$

λt	0.125	0.25	0.5	1	2	3
$P(t)$	0.88	0.78	0.61	0.37	0.14	0.05
$F(t)$	0.12	0.22	0.39	0.63	0.86	0.95

In the table, $P(t)$ is the probability that the event will occur within interval $[t; \infty]$; $F(t)$ is the probability that the event will occur within the interval $[0; t]$. If λ is the failure rate, then $P(t)$ is the probability of no-failure within interval $[0; t]$, while $F(t)$ is the probability of failure within interval $[0; t]$.

The first case is associated with the annual medical examinations that certain categories of workers undergo. Let us suppose that the time between two examinations is exponentially distributed with the average time of 1 year. Then, the rate of event "Medical examination" will be $\lambda = 1$ 1/year or $\lambda = 1/12$ 1/month. Let us set forth the predicted percentage (more specifically, the average percentage) for the following cases:

1) only 63% of workers will be examined within a year, while 37% will be examined in more than a year;

2) within 2 years, 14% of workers will not be examined;

3) within 3 years, 5% of workers will not be examined;

4) workers start undergoing examinations within the first months upon the previous examination; thus, within 3 months 22% will be examined, while within 6 months 39% will.

That pattern does not reflect the reality. Hence, the conclusion is that exponential distribution of time between preventive examinations is unacceptable.

The second case is associated with life expectancy. As it is known, the average life expectancy in Russia is 70 years (females live on average longer than males). Let us make calculations, assuming an exponential distribution of life expectancy of a person with the mean time of $m_t = 70$ years. The rate of event "end of life" is $\lambda = 1/70$ 1/year. The mean square deviation of a lifetime is $\sigma_t = 1/\lambda = 70$ years. Let us calculate the probability of event $0 \leq t \leq m_t + \sigma_t$, i.e., the probability of a person living from 0 to 140 years: $P(0 \leq t \leq m_t + \sigma_t) = P(0 \leq t \leq 140) = 0.86$. The probability of the event is $t > 140$: $P(t > 140) = 0.14$.

According to this calculation, an average 14% of people live to the age of 140 or more. Next, 5% of people live up to 210 years old. Everyone knows that no such people exist in Russia. Hence, the conclusion is that the assumption of exponential distribution of a person's life expectancy is

erroneous and must be rejected. This conclusion is based on personal experience and knowledge. If we did not have such knowledge, 14% would be accepted as a legitimate prediction. Thus, it can be concluded that the distribution of human life expectancy is not exponential. This distribution is to be substantiated using statistical methods.

The third case is associated with an 8-hour working day. Let us assume that due to the requirement of simulating a certain process, the assumption was made of exponential distribution of the working time with the rate of $\lambda = 1/8$ 1/hour. According to that assumption, 12% of workers will leave their workstations within an hour, while within 2 hours 22% of workers will do so. Next. Only 37% of workers will be on the job during 8 hours, while 14% will be on the job for 16 hours and 5% will be there an entire day (24 hours).

It is evident that the conclusions made on the assumption of exponential distribution of working time are implausible and even absurd.

The fourth case. Mathematical models of dependability often assume exponential distribution of the time between an item's technical state inspections. Let T_{mn} be the mean time between two inspections (the average period) under such distribution. Then, using the data from Table 1, the following can be concluded:

a) only 63% of inspections are conducted within the average period;

b) 14% and 5% of inspections are conducted within periods that exceed the average period two and three times, respectively;

c) 39% of inspections are conducted within a period twice shorter than the average one.

It can be assumed that an expert with a sufficient engineering experience will not accept such probability distribution associated with an item's technical state inspection.

A use case of unjustified exponential distribution

It should be noted that an unjustified use of exponential distribution may be viable as regards some (special) problems. Examples may include the case of two aircraft crashes published in [13].

In some countries, deadly plane crashes occur on average once a year. According to the media, two planes on different routes had crashed at a one-minute interval. The initial explanation of the disaster came down to technical issues (failures) of equipment. Let us do a probabilistic analysis of the situation. This analysis aims to explain the cause of the disasters, i.e., to confirm or deny the cause associated with technical issues (failures). In order to do that, let us assume an exponential distribution of the time between the disasters (with no due substantiation).

Let us denote as follows: A is the first plane crash; B is the second plane crash one minute after the first disaster. The probability of joint events, according to the multiplication theorem on probability, is: $p(AB) = p(A) \cdot p(B/A)$, where $p(B/a)$ is the conditional probability of the second

plane crash provided that the first one occurred. Obviously, $p(A) < 1$. Out of that follows that $p(AB) < p(B/A)$.

Let us calculate the conditional probability $p(B/A)$ under the assumption of exponential distribution of the time between the plane crashes. Under the conditions of the problem, the rate of disasters is $\lambda = 1$ 1/year. Let us convert the rate of disasters and the time between the two disasters into the same unit of time, namely, hours: $\lambda = 1/(365\cdot24)$ $= 1/h$; $t = 1$ min $= 1/60$ h. Let us calculate the product λt: $\lambda t = 1/(365\cdot24\cdot60) = 2\times10^{-6}$. The formula for calculating the conditional probability of the second plane crash, provided that the first one happened a minute before: $p(B/A) = 1 - e^{-\lambda t}$. Let us expand the exponential function into a power series, limiting ourselves to two terms: $e^{-\lambda t} \approx 1 - \lambda^t$. Let us calculate the conditional probability: $p(B/a) = p(B/A) = \lambda t = 2\times10^{-6}$. The probability of joint events can be estimated from the upper inequality: $p(AB) < 2\times10^{-6}$.

Thus, the upper bound of the probability of two joint plane crashes is obtained assuming an exponential distribution of the time between the disasters: 2×10^{-6}. This probability is close to zero. Therefore, the examined *AB* event should be considered almost impossible. In terms of the probability theory, the fact that this event did occur should be interpreted as follows: it can almost certainly be claimed that the two plane crashes did not occur by accident.

Notes. 1. The aim has been achieved. It was shown that the examined random event is practically impossible.

2. The problem can be solved using other distributions. Thus, if the time between disasters is distributed uniformly, the result is $p(AB) < 10^{-6}$. Both results are comparable and produce identical conclusions.

Background. On August 24, 2004, two airliners were attacked by suicide bombers. Airliners that departed from the Domodedovo airport crashed three minutes apart (Novaya Gazeta, 14.09.2011).

Case of substitution of a certain distribution with an exponential distribution

Let us provide an example of a distribution of time in a subset of states being substituted with an exponential distribution of such time. [18] considered continuous-time transitions between the operable, pre-failure and inoperable states. The operable state can transition into a pre-failure, while a pre-failure can transition into an inoperable state as the result of failure.
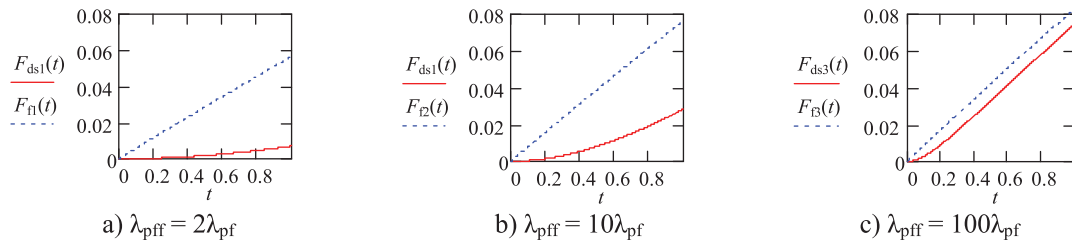
The probability of no-failure, or the probability of operable or pre-failure state with the initial operable state, obtained by solving differential equations:

$$P_{ds}(t) = \frac{\lambda_{pff} \cdot \exp\left(-\lambda_{pf} \cdot t\right) - \lambda_{pf} \cdot \exp\left(-\lambda_{pff} \cdot t\right)}{\lambda_{pff} - \lambda_{pf}}, \quad (1)$$

where λ_{pf} is the pre-failure rate; λ_{pff} is the rate of failures after pre-failures.

The distribution function of the time to failure

$$F_{ds}(t) = 1 - P_{ds}(t). \quad (2)$$

Within this model, the mean time to failure (or the mean time in the operable or pre-failure states) is

$$T_{ds} = \int_0^\infty P(t)dt = \frac{1}{\lambda_{pf}} + \frac{1}{\lambda_{pff}}. \quad (3)$$

Hence

$$T_{ds} = T_{pf} + T_{pff}, \quad (4)$$

where $T_{pf} = 1/\lambda_{pf}$ is the mean time before pre-failure; $T_{pff} = 1/\lambda_{pff}$ is the mean time between pre-failure and failure.

Formula (4) is presented in [1]. Additionally, the distribution of the time in this subset of states is substituted with an exponential distribution of time to failure with the rate of λ_f that is associated with the rates λ_{pf} and λ_{pff} and formula:

$$\frac{1}{\lambda_f} = \frac{1}{\lambda_{pf}} + \frac{1}{\lambda_{pff}}. \quad (5)$$

Thus, it was assumed that the time to failure with the rate λ_f is exponentially distributed. However, substituting distribution (1) with an exponential distribution requires substantiation. Out of (5) follows:

$$\lambda_f = \frac{\lambda_{pf} \cdot \lambda_{pff}}{\lambda_{pf} + \lambda_{pff}}, \quad (6)$$

while the probability of no-failure $P_f(t)$ and the time to failure distribution function $F_f(t)$ under an exponential distribution with the failure rate of λ_f are calculated using the following formulas:

$$P_f(t) = \exp(-\lambda_f \cdot t); \ F_f(t) = 1 - \exp(-\lambda_f \cdot t). \quad (7)$$

Let us consider the different relationships between the initial parameters within this model. Let us take the pre-failure rate $\lambda_{pf} = 10^{-5}$ 1 / h as a basis. For convenience, let us count time in years. For that purpose, let us represent the pre-failure rate as $\lambda_{pf} = 10^{-5}\cdot365\cdot24 = 0.0876$ 1/year.

Let us consider three types of relationships between λ_{pf} and λ_{pff}:
1) $\lambda_{pff} = 2\lambda_{pf}, \lambda_f = 2\lambda_{pf}/3$;
2) $\lambda_{pff} = 10\lambda_{pf}, \lambda_f = 10\lambda_{pf}/11$;
3) $\lambda_{pff} = 100\lambda_{pf}, \lambda_f = 100\lambda_{pf}/101$.

a) $\lambda_{pff} = 2\lambda_{pf}$ b) $\lambda_{pff} = 10\lambda_{pf}$ c) $\lambda_{pff} = 100\lambda_{pf}$

Fig. 1. $F_{ds}(t)$ and $F_f(t)$ distribution functions over the time period of 1 year under different relationships between the initial parameters

Fig. 1 a, b, c show the dependency graphs of the $F_{ds}(t)$ and $F_f(t)$ distribution function for these types respectively. The above graphs show that under the adopted relationships between the initial parameters, the $F_{ds}(t)$ and $F_f(t)$ distribution functions differ in every case.

It can be seen that the differences between $F_{ds}(t)$ and $F_f(t)$ decrease as the failure rate grows after pre-failures. This fact is referred to in [1]. Cable trunk operation data show that $T_{pf} >> T_{pff}$ is true in the overwhelming majority of cases. Then, $\lambda_f \approx \lambda_{pf}$.

It should be noted that the time period of 1 year in Fig. 1 suffices to conclude that calculating dependability indicators using $F_{ds}(t)$ and $F_f(t)$ will produce different results. Out of that follows that replacing the original process with an exponentially distributed process requires an error calculation.

Discussion

The paper considered examples of certain events and made a judgement on the applicability of exponential distribution. However, in case of such events as completion of recovery, duration of scheduled inspection, duration of maintenance, etc., a judgement regarding the applicability of exponential distribution cannot be made in the absence of personal experience associated with such events.

Similar conclusions can be made regarding the frequency of technical inspections of various equipment. For example, the time between verifications of water and electrical meters cannot be exponentially distributed, since the consumers will not significantly reduce the time between inspections, while companies will not allow long intervals between verifications. The real situation is that the time between inspections is still random. But it is not exponentially distributed. The distribution of such times is to be established using statistical methods.

[11] and [12] examined the models of operation of an item that is submitted to inspections with a constant period and with an exponentially distributed period. Those models were compared under the same constant period and average time between inspections. Formulas were obtained for calculating the availability coefficient, the non-availability coefficient and some other operational indicators. The calculated values of some indicators based on those models are identical, e.g., the average frequency of inspections, while some differ. Thus, there is a two-fold difference between the unavailability values for the above ways of defining the inspection frequency.

The use of exponential distribution or constant event rate (end of state) are to be clearly substantiated. Such substantiation may be based on the probability theory, mathematical statistics or otherwise.

Findings and conclusions

Thus, the above examples show that using exponential distribution for simulating random time between events is unacceptable after the semantic content of the example has been analysed.

The paper's findings allow making the following conclusions.

1. An unknown random distribution cannot be replaced with an exponential distribution without a valid substantiation. In other words, the use of exponential distribution as part of unknown distribution simulation is to be substantiated.

2. Replacing a random time in a subset of states with a random exponentially distributed time with a constant rate requires a valid substantiation.

3. Approximate calculations are to be provided with an error calculation.

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The author's contribution

The author analysed the application of the exponential distribution of time between states when used in mathematical models. This approach could be used to better substantiate the recommendations for applying the exponential distribution in matters of dependability.

Conflict of interests

The author declares the absence of a conflict of interests.