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METHOD FOR ELIMINATION OF MUTUAL CROSSING OF ESARY-PROSCHAN ESTIMATES IN TASKS OF ANALYSIS OF BIPOLAR NETWORKS CONNECTIVITY

The paper presents the method for elimination of mutual crossing and reduction of Esary-Proshan estimates errors.

Keywords: *probability of connection, probability of discontinuity, bipolar network, random graph of bipolar network, simple path, simple cut, upper bound, lower bound, exact value, Esary-Proshan estimates, bilateral estimates, elementary designs.*

1. Generalities and problem definition

Today the most known estimations of connectivity probability (CP) of bipolar networks (BN) are Esary-Proshan estimates (EPE) [1, 3, 7, 8, 9].

As estimated random graphs of bipolar networks (RG BN), EPE graphs offered in [3] use:

• For calculation of the upper bound of estimations – estimated RG BN, consisting of a full set of simple paths (SP) connected in parallel without the interdependence between circuits taken into account;

• For calculation of the lower bound of estimations – estimated RG BN, consisting of a full set of serially connected simple cuts (SC) without the interdependence between simple cuts taken into account.

However, the presence of dependences between SS or SC in corresponding estimated RG BN increases EPE errors in relation to the exact value of CP BN [2, 3].

In addition, EPE at incomplete use of a theoretically possible set of SP and SC in estimated RG BN mutually cross, that also being an essential disadvantage in EPE [3,6] (see fig. 6).

Owing to this, EPE retain "historical and methodical interest" but are not used for practical application [9].

Therefore, below we consider the method that allows us to eliminate mutual crossing and to receive monotonous convergence of EPE to the exact value of CP RG BN, and also to lower EPE errors.

Let us suppose that analyzed BN will be presented in the form of RG BN in which some message is transferred between vertices-poles S and t.

Let us believe that the mathematical model of RG BN is formally specified, if besides vertices-poles S and t, the restriction for the number of φ of transit vertices is also defined in connection paths between S and t as such that $0 \le \varphi \le (m_v-2)$, where $m_V = |V = \{\nu_{i=\overline{1,|V|}}\}|$ is the power of a vertex set RG BN.

For the accepted conditions, it is required to present a theoretical justification of the method providing monotonous (not overlapped) convergence of EPE to the exact value of CP RG BN, and also decrease of EPE errors.

2. The method's essence

In order to disclose the method's essence, we shall use some theoretical proposition in studies [10,11] and formulate the following theorem.

The theorem

Let the structure of analyzed RG BN consist of a vertex set $V=\{\vartheta_i\}, i=\overline{1,|m_V|}$, where $|m_V|$ is the power of a vertex set, and an edge set $L=\{l_\xi\},\ \xi=\overline{1,|m_L|}$, where $|m_L|$ is the power of an edge set, in whose structure there is a subset of shared (bridge) edges $L^0=\{l_\xi^0\},\ \xi=\overline{1,|m_{L^0}|},\ L^0\in L$. Let us assume that vertices of RG BN are absolutely dependable. Dependability of a bridge edge l_ξ^0 is described by two states: operable state l_ξ^0 or down state $\overline{l_\xi}$, with corresponding probabilities. In view of this condition, we shall define the number of all possible states (hypotheses) of bridge edges as:

$$\Gamma^0=2^{m_{L^0}}=\{\Gamma^0{}_i\},\ i=\overline{1,|m_{\Gamma o}|}, \tag{1}$$

where $|m_{\Gamma o}|$ is the power of a state set of shared edges.

Analyzed RG BN is characterized by SP set $\mathrm{M}=\{\mu_n\},\ \ \mathrm{n}=\overline{1,|\mathrm{m}_M|}$,, and SC set $\mathrm{R}=\{r_n\},\mathrm{n}=\overline{1,|\mathrm{m}_R|},|\mathrm{m}_R|$, where $|\mathrm{m}_M|$ is the power of SP set, and $|\mathrm{m}_R|$ is the power of SC set, which accordingly form the structure of estimated RG BN under SP – $G_{s,t}^{ПЦ}$ and the structure of estimated RG BN under SC – $G_{s,t}^{ПР}$.

Estimated RG BN $G_{s,t}^{ПЦ}$ and $G_{s,t}^{ПР}$ when finding a subset of their bridge edges $L^0=\{l_\xi^0\}$ in state Γ_i^o will be transformed accordingly into conditional estimated RG BN $G_{s,t}^{ПЦ}$ and $G_{s,t}^{ПР}$, i.e.

$$\begin{cases}\mathrm{G}_{s,t}^{ПЦ}/\Gamma_i=G_{s,t(i)}^{ПЦ}\\ \mathrm{G}_{s,t}^{ПР}/\Gamma_i=G_{s,t(i)}^{ПР}\end{cases}. \tag{2}$$

In studies [3,4] it was shown that in order to calculate the upper bound of EPE, the following equality should be applied:

$$P_{s,t}^{BO}=1-P_{\overline{s,t}}^{HO}, \tag{3}$$

where $P_{\overline{s,t}}^{HO}$ is the lower bound of discontinuity probability of vertices s and t in RG BN.

For the specified conditions, it is required to prove the validity of application formulas of the following form for calculation of CP upper and lower bounds of analyzed RG BN:

$$P_{s,t}^{BO}=1-\left[P_{\overline{s,t}}^{HO}=\sum_{i=1}^{m_{\Gamma^o}}P_{\overline{s,t_i}}^{HO}=\sum_{i=1}^{m_{\Gamma^o}}P\left(\Gamma^0{}_i\right)\overline{P\left(G_{s,t_i}^{ПЦ}/\Gamma^0{}_i\right)}\right], \tag{T. 1}$$

$$P_{s,t}^{HO}=\sum_{i=1}^{m_{\Gamma^o}}P_{s,t_i}^{HO}=\sum_{i=1}^{m_{\Gamma^o}}P\left(\Gamma^0{}_i\right)P\left(G_{s,t_i}^{ПР}/\Gamma^0{}_i\right), \tag{T. 2}$$

where $P(\Gamma^0{}_i)$ is the probability of the i-th state of a subset of bridge edges $L^0=\{l_\xi^0\},\xi=\overline{1,|m_{L^0}|}$;

$\overline{P\left(G^{ПЦ}_{s,t} / \Gamma^{0}_{i}\right)}$ is the conditional probability of discontinuity of vertices S and t of RG BN under SP, corresponding to hypothesis Γ^{0}_{i}. For the sake of brevity of notation, we shall designate this probability as P $\left(G^{ПЦ}_{s,t_i}\right)$, where $G^{ПЦ}_{i} = G^{ПЦ}_{s,t_i} / \Gamma^{0}_{i}$;

$P\left(G^{ПР}_{s,t} / \Gamma^{0}_{i}\right)$ is the conditional probability of connection of vertices S and t RG BN under SC, corresponding to hypothesis Γ^{0}_{i}. This estimation shall be designated as $P\left(G^{ПР}_{s,t_i}\right)$, where $G^{ПР}_{s,t_i} = G^{ПР}_{s,t} / \Gamma^{0}_{i}$;

$$P^{HO}_{s,t_i} = P\left(\Gamma^{0}_{i}\right)\overline{P\left(G^{ПЦ}_{s,t} / \Gamma^{0}_{i}\right)} \tag{T.3}$$

(T3) is the probability of discontinuity of vertices S and t of analyzed RG BN under SP provided that RG BN is in state Γ^{0}_{i}.

The proof

As the set of hypotheses (states) $\Gamma^{0}_{i} = \{\Gamma^{0}_{i}\}, i = \overline{1, m_{\Gamma^0}}$ of a subset of bridge edges $L^0 = \{l^{0}_{\xi}\}, \xi = \overline{1, m_{L^0}}$, form a full group of disjoint events, and conditional RG BN (2) can be formed only according to one of the hypotheses $\Gamma^{0}_{i}, i = \overline{1, m_{\Gamma}}$, then according to the contents of the formula for full probability [2], equality (T.1) and (T. 2) are determined correctly. The theorem has been proved.

The consequence

For $\xi = 1,\ \Gamma^0 = 2^1 = 2:\ \Gamma^{0}_{1} = l_1,\ \Gamma^{0}_{2} = \overline{l}_1$.
Then:

$$P^{BO}_{s,t} = 1 - \left[P^{HO}_{\overline{s,t}} = \sum_{i=1}^{2} P^{HO}_{\overline{s,t_i}} = \sum_{i=1}^{2} P\left(\Gamma^{0}_{i}\right)\overline{P\left(G^{ПЦ}_{s,t_i} / \Gamma^{0}_{i}\right)}\right], \tag{4}$$

$$\text{where } P^{HO}_{\overline{s,t_i}} = P(\Gamma^{0}_{i})P(G^{ПЦ}_{s,t} / \Gamma^{0}_{i}), \tag{5}$$

$$P^{HO}_{s,t} = \sum_{i=1}^{2} P^{HO}_{s,t_i} = \sum_{i=1}^{2} P\left(\Gamma^{0}_{i}\right)P\left(G^{ПР}_{s,t_i} / \Gamma^{0}_{i}\right), \tag{6}$$

At $\xi = 2,\ \Gamma^0 = 2^2 = 4:\ \Gamma^{0}_{1} = l_1 \cdot l_2;\ \Gamma^{0}_{2} = \overline{l}_1 \cdot l_2;\ \Gamma^{0}_{3} = l_1 \cdot \overline{l}_2;\ \Gamma^{0}_{4} = \overline{l}_1 \cdot \overline{l}_2.$

For this alternative:

$$P^{BO}_{s,t} = 1 - \left[P^{HO}_{\overline{s,t}} = \sum_{i=1}^{4} P^{HO}_{\overline{s,t_i}} = \sum_{i=1}^{4} P\left(\Gamma^{0}_{i}\right)\overline{P\left(G^{ПЦ}_{s,t_i} / \Gamma^{0}_{i}\right)}\right], \tag{7}$$

$$\text{where } P^{HO}_{s,t} = \sum_{i=1}^{4} P^{HO}_{s,t_i} = \sum_{i=1}^{4} P\left(\Gamma^{0}_{i}\right)P\left(G^{ПР}_{s,t_i} / \Gamma^{0}_{i}\right), \tag{8}$$

etc.
The probability of hypothesis Γ^{0}_{i} we shall determine as follows:

$$P\left(\Gamma^{0}_{i}\right) = \prod_{\xi \in \beta_1} P\left(l^{0}_{\xi}\right) \prod_{\xi \in \beta_2} q\left(l^{0}_{\xi}\right), \tag{9}$$

where β_1 and β_2 are designations according to subsets of operable and failed bridge edges forming hypothesis Γ_i^0;

$P\left(l_\xi^0\right)$ is the probability of finding the shared edge l_ξ^0 in operable state, $q\left(l_\xi^0\right)=1-P\left(l_\xi^0\right)$.

Calculation of conditional probabilities $P(G_{s,t_i}^{ПЦ})$ and $P(G_{s,t_i}^{ПР})$ is also feasible according to the structures of conditional estimated RG BN under SP – $G_{s,t_i}^{ПЦ}$ and under SC – $G_{s,t_i}^{ПР}$, obtained accordingly from estimated RG BN $G_{s,t}^{ПЦ}$ and $G_{s,t}^{ПР}$ provided that the subset $L^0=\left\{l_\xi^0\right\}$ of shared edges is in Γ_i state.

Let us show that the application of a combinatorial method eliminates mutual crossing of EPE. With this purpose we shall formulate and prove the following statement.

The statement

It is given: initial RG BN (fig. 1) and its estimated RG BN (fig. 2a and fig. 2b) for calculation of the upper estimate $P_{s,t}^{BO}$ and the lower estimate $P_{s,t}^{HO}$ of connection probability accordingly. If the offered combinatory method is applied to calculation of $P_{s,t}^{BO}$ and $P_{s,t}^{HO}$, then bilateral estimations of CP received at the i-th iterative steps do not overlap.

We need to prove:

$$P_{s,t}^{BO}\left(\Gamma_i^0\right)>P_{s,t}^{HO}(\Gamma_i^0), \tag{U. 1}$$

for all values $0<p(l_\xi)<1$.

The proof

As all the possible hypotheses $\Gamma^0=\{\Gamma_i^0\}$ used for definition of the upper and lower bounds of EPE are identical and make up a full group, then according to a Moore-Shannon decomposition formula the inequality $P_{s,t}^{BO}\left(\Gamma_i^0\right)>P_{s,t}^{HO}(\Gamma_i^0)$ is true for each i-th hypothesis for all values $0<p(l_\xi)<1$. The statement has been proved.

To illustrate the content of the offered method for analysis of bilateral EPE, we shall use two-bridge RG BN with continuous numbering of elements, fig. 1 [3].

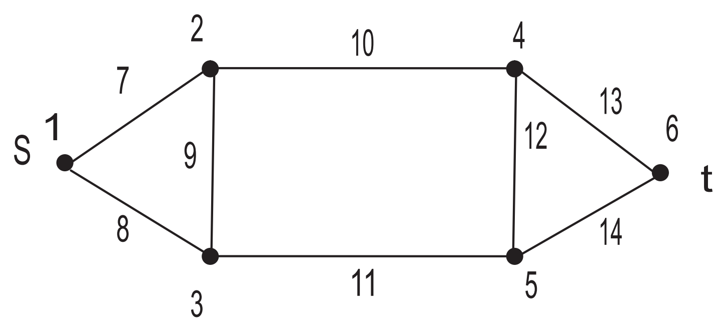

Fig. 1. Two-bridge RG BN with continuous numbering of elements

Structural elements in fig. 1 have only digital designations. This RG BN is characterized by a set of SP (fig. 2, a) of the following form:

$$M_{s,t}=\begin{Bmatrix} M_{n=1}=(7,10,13), M_{n=2}=(8,11,14), M_{n=3}=(7,10,12,14), M_{n=4}=(7,9,11,14), \\ M_{n=5}=(8,9,10,13),\ M_{n=6}=(8,11,12,13),\ M_{n=7}=(7,9,11,12,13),\ M_{n=8}=(8,9,10,12,14) \end{Bmatrix}, \tag{10}$$

determining the upper bound of EPE and also by a set of SC (fig. 2, b) having the following form:

$$R_{s,t}=\left\{\begin{matrix} r_{n=1}=(7,8), r_{n=2}=(10,11), r_{n=3}=(13,14), r_{n=4}=(8,9,10),\ r_{n=5}=(11,12,13), \\ r_{n=6}=(7,9,11),\ r_{n=7}=(10,12,14),\ r_{n=8}=(8,9,12,13),\ r_{n=9}=(7,9,12,14) \end{matrix}\right\}, \qquad (11)$$

determining the lower bound of EPE.

Bridge edges 9 and 12 presented in fig. 2, a and 2, b are marked by thin double lines.

As the structure of initial RG BN (fig. 1) has two bridge edges (9 and 12), then the full group of disjoint states (hypotheses) of these edges will take the form:

$$\Gamma^0=\left\{\Gamma_1^0=(9\cdot 12);\ \Gamma_2^0=(\overline{9}\cdot 12);\ \Gamma_3^0=(9\cdot \overline{12});\ \Gamma_4^0=(\overline{9}\cdot \overline{12})\right\}. \qquad (12)$$

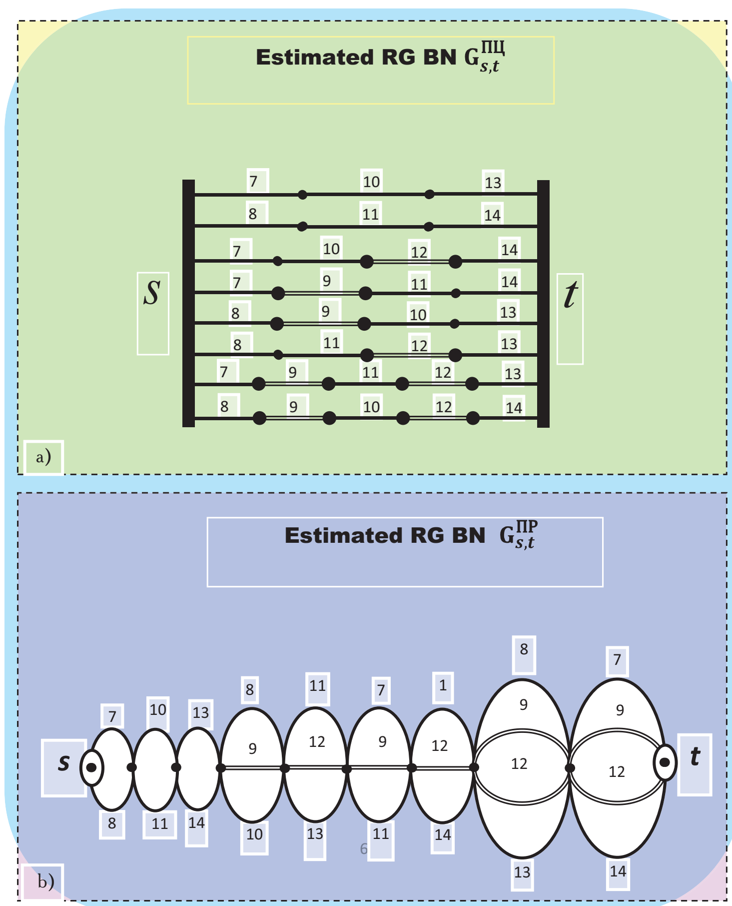

Fig. 2. Diagrams of estimated RG BN for EPE: a) under ПЦ; b) under ПР

In view of the full group of disjoint states (10), transformation of initial estimated RG BN $G_{s,t}^{ПЦ}$ (fig. 2, a) in conditional estimated RG BN $G_{s,t_i}^{ПЦ}$, presented in fig. 3 was carried out. And also, transformation of initial estimated RG BN $G_{s,t}^{ПЦ}$ (fig. 2, b) in conditional estimated RG BN $G_{s,t_i}^{ПЦ}$, presented in fig. 4 was made.

On presented conditional RG BN (fig. 3 and fig. 4) dark circles mean connection of adjacent vertices in the shared point over special edges 9 and 12 when these edges are in upstate. If these edges are in down state, then breaks are formed in corresponding elementary constructions (SP and SC) which in the mentioned above figures are represented by dashed lines.

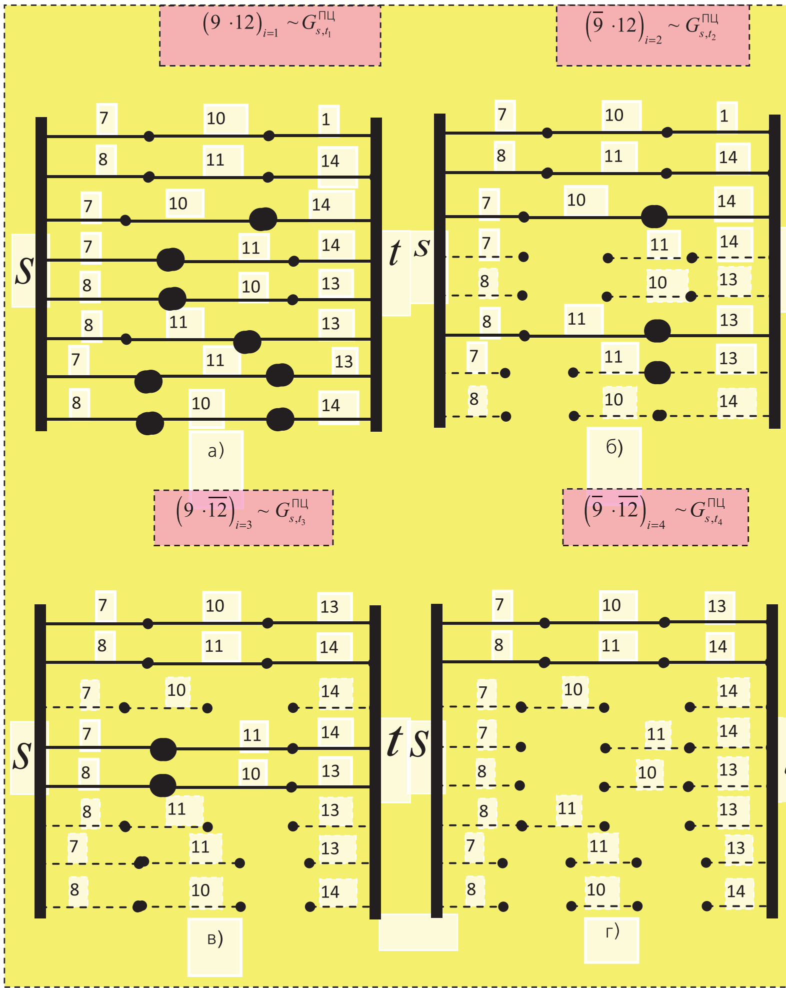

Fig. 3. Transformation of estimated RG BN $G^{ПЦ}_{s,t}$ at acceptance of the i-th hypotheses

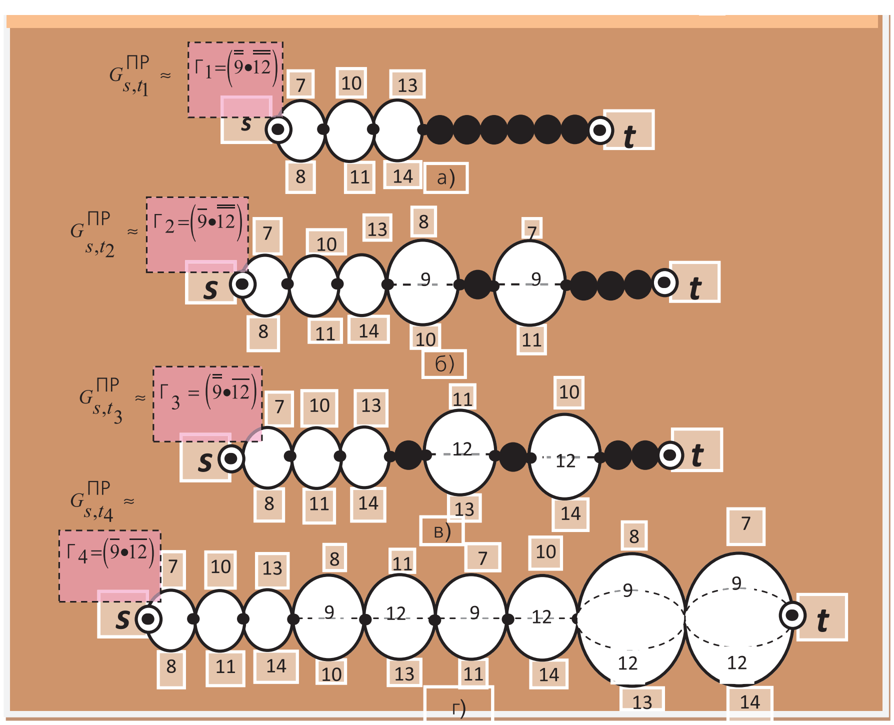

Fig. 4. Transformation of estimated RG BN $G^{ПР}_{s,t}$ at acceptance of the i-th hypotheses

The formulated and proved statements concerning validity of a combinatory method application for elimination of EPE mutual overlapping shall be confirmed by the following computing experiment.

Let us calculate upper $P_{s,t}^{BO}$ and lower $P_{s,t}^{HO}$ bounds of CP RG BN (fig. 1) for the following initial data (ID) on dependability of edges $p_{\xi=\overline{7,14}}=0,5$.

For accepted ID, estimation (5), in view of equality (7), shall be determined in the following form:

$$P_{\underline{s,t}}^{HO}=0_{i=0}+0,085902228_{\underline{i=\bar{1}}}+0,14654541_{i=2}+0,14654541_{i=3}+0,19140625_{i=4}=0,570399298. \quad (13)$$

Then, according to (13), we shall determine monotonous convergence $P_{s,t}^{BO}$ to $P_{s,t}$ as follows:

$$P_{s,t}^{BO}\to P_{s,t}=\underline{\underline{1}}_{i=0}\to\left(1-0,085902228_{i=\overline{0,1}}=\underline{\underline{0,914097772}}\right)_{i=\overline{0,1}}\to$$

$$\to\left(1-0,232447638_{i=\overline{0,2}}=\underline{\underline{0,767552365}}\right)_{i=\overline{0,2}}\to\left(1-0,378993048_{i=\overline{0,3}}=\right.$$

$$\left.=\underline{\underline{0,621006952}}\right)_{i=\overline{0,3}}\to\left(1-0,570399298_{i=0.4}=\underline{\underline{0,429600702}}=P_{s,t}^{BO}\right)_{i=\overline{0,4}}. \quad (14)$$

For accepted ID $p_{\zeta=\overline{7,14}}=0,5=q_{\zeta=\overline{7,14}}$ the lower bound CP of analyzed RG BN (Fig. 1) according to (8) will be equal to:

$$P_{s,t}^{HO}=0_{i=0}+0,10546875_{i=\bar{1}}+0,059326171_{i=2}+0,059326172_{i=3}+0,018771171_{i=4}=$$

$$=0_{i=0}\to\underline{\underline{0,10546875}}_{i=1}\to\underline{\underline{0,164794921}}_{i=\overline{1,2}}\to\underline{\underline{0,224121092}}_{i=\overline{1,3}}\to\underline{\underline{0,242892263}}_{i=\overline{1,4}}. \quad (15)$$

On the basis of equality (14) and (15), fig. 5 presents graphs of EPE monotonous convergence to the exact value $P_{s,t}^{HO}\to P_{s,t}\leftarrow P_{s,t}^{BO}$ for initial RG BN, fig. 1. In fig. 5 "asterisk" shows step-by-step convergence of the following error $P_{s,t_i}\approx(P_{s,t_i}^{\mathrm{BO}}+P_{s,t_i}^{\mathrm{HO}})/2$ to the exact value $P_{s,t}$.

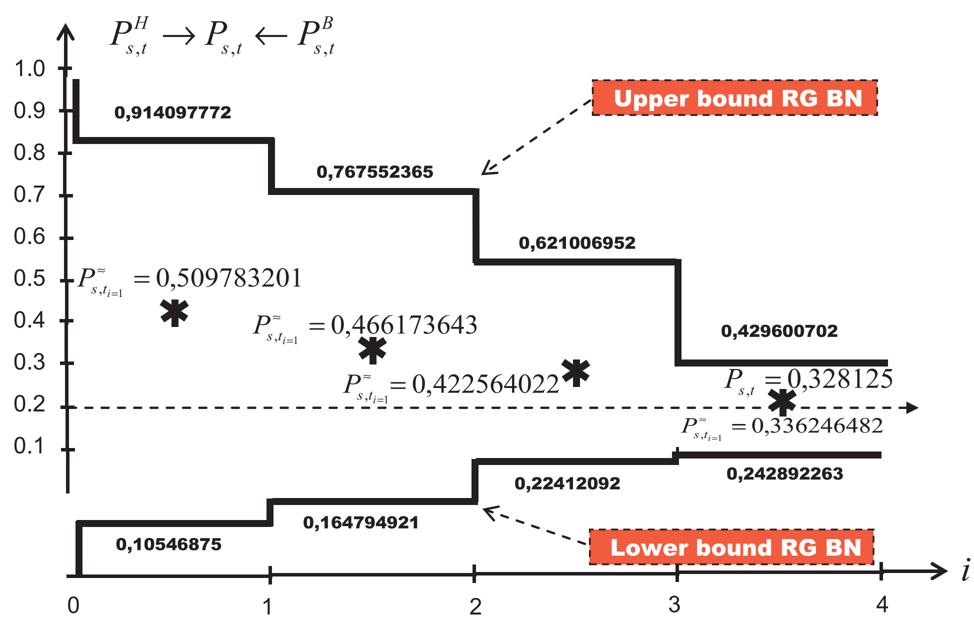

Fig. 5. Illustration of EPE monotonous convergence to the exact value

Fig. 6 presents graphic interpretation of basic EPE in dynamics of their step-by-step change, with regard to RG BN analysis (fig. 1) and ID of the following kind $P_{\xi=\overline{7,14}} = 0,5 = q_{\xi=\overline{7,14}}$ [3].

The analysis of the results of the executed numerical experiment presented in fig. 5 and their comparison with basic EPE (fig. 6) shows that the offered combinatory method of the analysis provides elimination of mutual crossing and reduction of CP RG BN errors in relation to basic EPE.

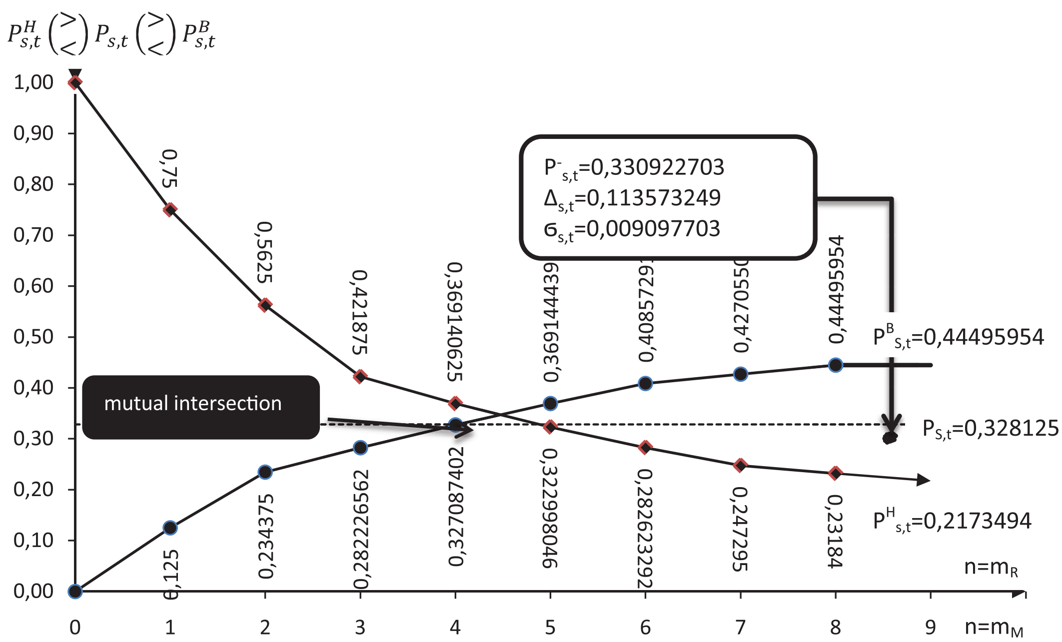

Fig. 6. Illustration of basic EPE crossing

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