Original article
https://doi.org/10.21683/1729-2646-2021-21-1-11-16

# A method of using the transitive graph of a Markovian process as part of ranking of heterogeneous items

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**Abstract.** *Hierarchy analysis developed by Thomas Saaty is a closed logical structure that uses simple and well-substantiated rules that allow solving multicriterial problems that include both quantitative, and qualitative factors, whereby the quantitative factors can differ in terms of their dimensionality. The method is based on problem decomposition and its representation as a hierarchical arrangement, which allows including into such hierarchy all the knowledge the decision-maker has regarding the problem at hand and subsequent processing of decision-makers' judgements. As the result, the relative degree of the interaction between the elements of such hierarchy can be identified and later quantified. Hierarchy analysis includes the procedure of multiple judgement synthesis, criteria priority definition and rating of the compared alternatives. The method's significant limitation consists in the requirement of coherence of pairwise comparison matrices for correct definition of the weights of compared alternatives. The* **Aim** *of the paper is to examine a non-conventional method of solving the problem of alternative ratings estimation based on their pairwise comparisons that arises in the process of expert preference analysis in various fields of research. Approaches are discussed to the generation of pairwise comparison matrices taking into consideration the problem of coherence of such matrices and expert competence estimation.* **Method.** *The methods of hierarchy analysis, models and methods of the Markovian process theory were used.* **Result.** *The paper suggested a method of using the transitive graph of a Markovian process as part of expert ranking of items of a certain parent entity subject to the competence and qualification of the experts involved in the pairwise comparison. It is proposed to use steady-state probabilities of a Markovian process as the correlation of priorities (weights) of the compared items. The paper sets forth an algorithm for constructing the final scale of comparison taking into consideration the experts' level of competence.* **Conclusion.** *The decision procedures, in which the experts are expected to choose the best alternatives out of the allowable set, are quite frequently used in a variety of fields for the purpose of estimation and objective priority definition, etc. The described method can be applied not only for comparing items, but also for solving more complicated problems of expert group estimation, i.e., planning and management, prediction, etc. The use of the method contributes to the objectivity of analysis, when comparing alternatives, taking into consideration various aspects of their consequences, as well as the decision-maker's attitude to such consequences. The suggested model-based approach allows the decision-maker identifying and adjusting his/her preferences and, consequently, choosing the decisions according to such preferences, avoiding logical errors in long and complex reasoning chains. This approach can be used in group decision-making, description of the procedures that compensate a specific expert's insufficient knowledge by using information provided by the other experts.*

**Keywords:** *Markovian process, expert assessment, transitive graph, ranking, incomplete comparisons.*

**For citation:** *Bochkov A.V. A method of using the transitive graph of a Markovian process as part of ranking of heterogeneous items. Dependability 2021;1: 11-16. https://doi.org/10.21683/1729-2646-2021-21-1-11-16.*

**Received on:** *10.12.2020* / **Upon revision:** *04.02.2021* / **For printing:** *22.03.2021*

## Introduction

In many cases, problems that require expert evaluation imply the involvement of many experts, which, in turn, imposes special requirements on the procedure of integration of estimates and their reliability. In most decision-making tasks there are procedures that allow combining the opinions of several experts regarding the alternatives presented to them [1, 2, 3]. As each expert has a unique experience, the opinions of various experts may significantly differ (indeed, there are many factors that affect an expert's preferences). This variety of expert assessments may cause a situation where some of them are unable to adequately express any degrees of preference by comparing two or more available alternatives. Additionally, the need arises to use the opinions of various experts, which causes another serious source of estimate inconsistency, i.e., the effect of disagreement between the experts' opinions. Sometimes, in a bid to ensure the agreement of comparison, estimates that significantly differ from average are discarded, but Little and Millet [4, 5], as well as Nogin [6] have shown that eventually that may cause the loss of influential common factors. In some cases, the expert polling procedure can be modified [10, 11]. The applicability and efficiency of individual approaches, in principle, depends on the number of gaps in data and reasons of their occurrence [12]. However, Carmone et al. [13], using a specific example, have shown that a "random removal of up to 50 percent of comparisons provides good results with no loss of accuracy". In this case, generalizing the results is incorrect; they rely on a priori knowledge of the complete matrix of pairwise comparison, which is practically impossible. There are approaches that, provided the availability of an incomplete matrix of pairwise comparison, enable methods that allow completing the matrix, which is confirmed by a number of researchers [14]. A system that helps build fuzzy preference relations was suggested in [15, 16]; a similar method is also described in Russian sources [17]. Group decision-making, description of procedures that correct the absence of knowledge in a specific expert using information provided by the other experts, along with some aggregation procedures can be found in [18 – 20].

## 1. Problem definition

The complexity of comparative estimation (ranking) of heterogeneous items of a certain set consists in the fact that each item is normally characterized by not one indicator (process-specific characteristics, certain intrinsic attributes, factors, pricing parameters, etc.), but several ones, whereby such indicators often vary in their nature.

In the general case, it is required to arrange all the items $x_k$ ($k$=1, …, $n$) of a certain set $G$ or define the rank (weight) of such items in set $G$, i.e., define preference correlations between the items $x^1 \succ x^2 \succ ... \succ x^n$.

Due to the insufficient amount of information on the compared items, the ranking problem is often solved with the involvement of experts that, upon a meticulous informal analysis, specify an order of preference for such items.

In this case, when the number of comparison parameters is high, it is very difficult for an expert to analyze them, which causes the requirement for item ranking methods that do not require significant calculations and are based on much simpler expert estimates. Such methods are to allow "combining" often contradictory pairwise comparative estimates of experts.

## 2. Method of solution

Let us briefly describe the essence of the suggested method of processing of expert estimates. A pairwise comparison matrix can be represented in the form of a connected graph with $N$ nodes and two edges between a pair of nodes, with one characterizing the transition from the $i$-th state into the $j$-th state at the rate of $\varepsilon_{ij}$, while the other one characterizing the transition from the $j$-th state into the $i$-th at the rate of $\varepsilon_{ji}$. For instance, if two items compared, and the weights of items no. 1 and no. 2, in the first expert's opinion, are equal to 1 and 2, respectively, the Markovian network constructed for such case is shown in Fig. 1.

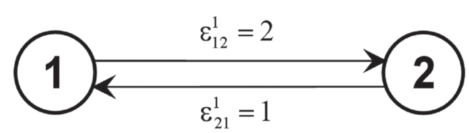


Fig. 1. Markovian process of comparing two items

The final state probabilities are equal to, respectively, $p_1^1 = \frac{\varepsilon_{21}^1}{\varepsilon_{12}^1 + \varepsilon_{21}^1} = \frac{1}{3}$ and $p_2^1 = \frac{\varepsilon_{12}^1}{\varepsilon_{12}^1 + \varepsilon_{21}^1} = \frac{2}{3}$, while their correlation is 1:2. This simple reasoning allows us, out of a paired preference, finding the transition rates for such graphs: $\frac{p_1^1}{p_1^2} = \frac{\varepsilon_{21}^1}{\varepsilon_{12}^1}$. The values of $\varepsilon_{ij}$ are chosen subject to condition $\sum_{i,j=1}^{n} \varepsilon_{ij} + \varepsilon_{ji} = 1$.

Let us continue our reasoning. Let us assume that the second expert estimated the rates for the same pair of items at $\varepsilon_{21}^2 = 3$ and $\varepsilon_{12}^2 = 1$ respectively. If both experts have identical "confidence coefficients" ("weights"), i.e., $\omega^1=\omega^2=\omega=0{,}5$, then, obviously, the "compromise" rates will equal $\varepsilon_{21}^* = \omega^1 \cdot \varepsilon_{21}^1 + \omega^2 \cdot \varepsilon_{21}^2 = 0{,}5 \cdot \varepsilon_{21}^1 + 0{,}5 \cdot \varepsilon_{21}^2$ and $\varepsilon_{12}^* = 0{,}5 \cdot \varepsilon_{12}^1 + 0{,}5 \cdot \varepsilon_{12}^2$. Generalizing the above reasoning for a larger number of experts and accounting for their "weights" is trivial.

After the design has been constructed, a standard final probabilities calculation is conducted for process states [21, p. 404].

In order to calculate the final probabilities, Kolmogorov-Chapman equations can be used as well:

$$\begin{cases} \frac{dp_1}{dt} = \left(\varepsilon_{12} p_2 + ... + \varepsilon_{1n} p_n\right) - \left(\varepsilon_{21} p_1 + ... + \varepsilon_{n1} p_1\right), \\ ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ... \; ..., \\ \frac{dp_n}{dt} = \left(\varepsilon_{n1} p_1 + ... + \varepsilon_{nn-1} p_{n-1}\right) - \left(\varepsilon_{1n} p_n + ... + \varepsilon_{n-1n} p_n\right). \end{cases}$$

Let us note that, provided all pairwise estimates are available, comparing $N$ items (i.e., ranking on the number axis) only requires compliance with the requirement of the transitive nature of the transition graph of the Markovian process. An example of such graph is sown in Fig. 2.

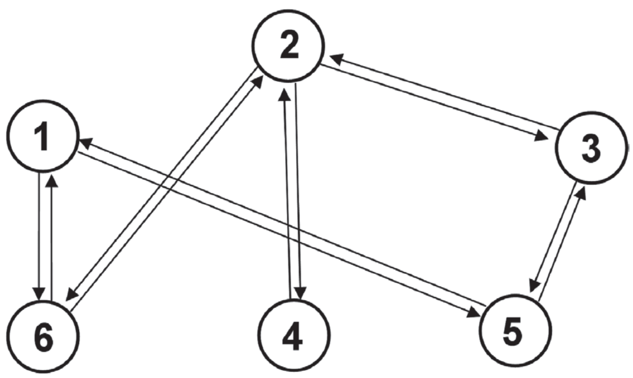


Fig. 2. Transitive graph of a Markovian process for the case of six items estimation

For the sake of specificity, let us examine the case of ranking of six items. Three experts were polled. Let each expert do all the possible pairwise comparisons of items. For each expert, according to the above method, a complete graph of a Markovian process can be constructed and all steady-state probabilities of states can be calculated. Let us assume that such probabilities will be the "weights" of the items defined by the experts. Then, the final probabilities are calculated for the "integral" graph, where the transition rates are just the average of all the rates (the "weights" of experts are assumed to be $1/N$, $N$ being the number of experts).

## 3. Estimation of the degree of expert incompetence

Thus, let us examine a case of estimation of six items ($i$=1, …, 6) by three experts ($k$=1, 2, 3). The results of estimation (final probabilities $p_i^k$) are shown in Table 1.

**Table 1. Estimation of six items by a group of three experts**

| Name of item | Experts | | |
|---|---|---|---|
| | 1 | 2 | 3 |
| Item no. 1 | 1 | 2 | 3 |
| Item no. 2 | 2 | 1 | 1 |
| Item no. 3 | 5 | 4 | 5 |
| Item no. 4 | 4 | 5 | 4 |
| Item no. 5 | 3 | 3 | 6 |
| Item no. 6 | 6 | 6 | 3 |

Calculating the corresponding average values of expert estimates:

$$\overline{p}^k = \frac{1}{6} \sum_{i=1}^{6} p_i^k. \qquad (1)$$

The dispersion of expert estimates is, respectively:

$$D^1 = \frac{1}{6} \sum_{i=1}^{6} \left( \frac{p_i^1 - \overline{p}^1}{\overline{p}^1} \right)^2,$$
$$D^2 = \frac{1}{6} \sum_{i=1}^{6} \left( \frac{p_i^2 - \overline{p}^2}{\overline{p}^2} \right)^2, \qquad (2)$$
$$D^3 = \frac{1}{6} \sum_{i=1}^{6} \left( \frac{p_i^3 - \overline{p}^3}{\overline{p}^3} \right)^2.$$

As the experts use different scales for the purpose of estimation, some relative values are allowable. The object of further research is the estimate of the measure of spread for non-significant and significant items, as the experts may have the corresponding "specialism". In other words, the confidence in the experts (their final "weights") may depend on the item category.

The following procedure is suggested:

a) for each expert, individual "coordinated[1] weights" of items are found from the graph with specified rates ($p_i^k$);

b) the same operation is repeated for a model, where all transition rates are equal to the average values of the respective rates for all experts ($\overline{p}^k$);

c) for each expert, the following value is calculated:

$$\delta^k = \sum_{1 \le k \le n} (p_i^k - \overline{p}^k)^2. \qquad (3)$$

The higher is the value, the lower is the confidence in the expert's opinion.

## 4. Definition of expert weights

Given (3), the adjusted "weights" of experts can be found according to formula

$$w^k = \frac{1 - \delta^k}{6 - \sum_{1 \le j \le n} \delta^j}. \qquad (4)$$

Let us plot a random graph with rates on the edges of, respectively

$$\overline{\varepsilon}_{ij} = \sum_{1 \le k \le 6} w^k \varepsilon_{ij}^k. \qquad (5)$$

[1] The weights are called coordinated, as the same expert may provide contradictory pairwise estimates, e.g., $A > B$, $B > C$, but $C > A$.

**Table 2. Item estimates**

| Name of item | Experts | | | Final weight of items | Rank of items |
|---|---|---|---|---|---|
| | 1 | 2 | 3 | | |
| Item no. 1 | 1 | 2 | 3 | 2.0865 | 2 |
| Item no. 2 | 2 | 1 | 1 | 1.4398 | 1 |
| Item no. 3 | 5 | 4 | 5 | 4.9662 | 5 |
| Item no. 4 | 4 | 5 | 4 | 4.6429 | 4 |
| Item no. 5 | 3 | 3 | 6 | 4.1729 | 3 |
| Item no. 6 | 6 | 6 | 3 | 5.4361 | 6 |
| Average estimate | 3.50 | 3.50 | 3.67 | | |
| Dispersion | 0.2380952 | 0.2380952 | 0.1900826 | | |
| Sigma | 17.50 | 17.50 | 15.33 | | |
| Weight of expert | 0.3721805 | 0.3721805 | 0.3233083 | | |

The results of calculations per formulas (1) to (5) are shown in Table 2.

Further, the above algorithm repeats: the state probabilities, dispersions and new "weights" of experts are calculated. At the same time, a procedure should be provided to eliminate the experts that show a dispersion of estimates above a certain threshold (in other words, incompetent in the subject matter).

## 5. Final scale construction

For the purpose of building the final scale, let us use the procedure that is a modified algorithm of the Delphi method[1]. In general, the Delphi algorithm is a process, as the result of which the group members (independent experts) come to a consensus in relation to certain events with no face-to-face discussions. This contributes to independent thinking on part of the group members, prevents direct confrontation between the participants and does not allow them defending their opinion by imposing their views upon other experts of the group. Importantly, looking for a solution using this method allows taking into consideration minority opinion, whereby in individual cases it may play a decisive role.

This method is the most formal out of all methods of expert prediction and is most often used in technological forecasting, of which the data are later used in the planning of products manufacture and sales. This is a group method, whereas a group of experts is individually polled regarding their assumptions on future events in various areas, where discoveries or improvements are expected. Polling is done with the use of special questionnaires anonymously (personal contacts between experts and collective discussions are not allowed).

The obtained answers are summarized and the results are sent back to group members. For that purpose, the average and weighted average values of the examined parameter are calculated, the median is defined as the medium term of the general series of estimates obtained from the experts and the confidence area[2].

Based on that information, the group members (while remaining anonymous) express further assumptions, whereby this process may repeat several times as part of the so-called multi-tour polling procedure.

The experts can be conventionally divided into three categories:

– the "conservatives" who do not change their estimates;

– the "listeners" who start changing their estimate by bringing it closer to the average one;

– the "stubborn" who assign estimates that are further off the average one.

The results of expert polling are used as the prediction as the first matching opinions appear.

In general terms, the procedure of expert polling based on the Delphi method includes the following five stages.

Stage 1. Formation of the working group of analysts tasked with organizing expert polling.

Stage 2. Formation of the expert group. In accordance with the method's requirements, the group of experts is to include 10 to 15 subject matter experts. The experts' competence is defined by means of a questionnaire survey, referencing analysis (number of reference to the expert's publications), self-rating sheets.

Stage 3. Question definition. The wordings of the questions must be clear and unambiguous, imply clear-cut answers.

Stage 4. Expert evaluation. The method implies polling in a number of steps.

Stage 5. Conclusion.

The suggested algorithm of final scale construction is similar to the above algorithm of expert evaluation based on the Delphi method (Fig. 3).

[1] Another name of the method is the Delphian oracle method. Authors of the method: RAND mathematicians O. Helmer, T. Gordon T. and others (US), 1950s.

[2] The confidence area should be calculated through the quartile (term first used by Galton, 1882). There are three dividing points, i.e. the lower, medium and upper quartiles (also called quantiles 0.25; 0.5 and 0.75) that equal the 25-th, 50-th and 75-th percentiles of distribution (respectively). The 25-th percentile of a variable is such value, below which fall 25% of such variable's values. Similarly, the 75-th percentile is such value, below which fall 75% values of the variable. The medium quartile (50-th percentile) is called the median.

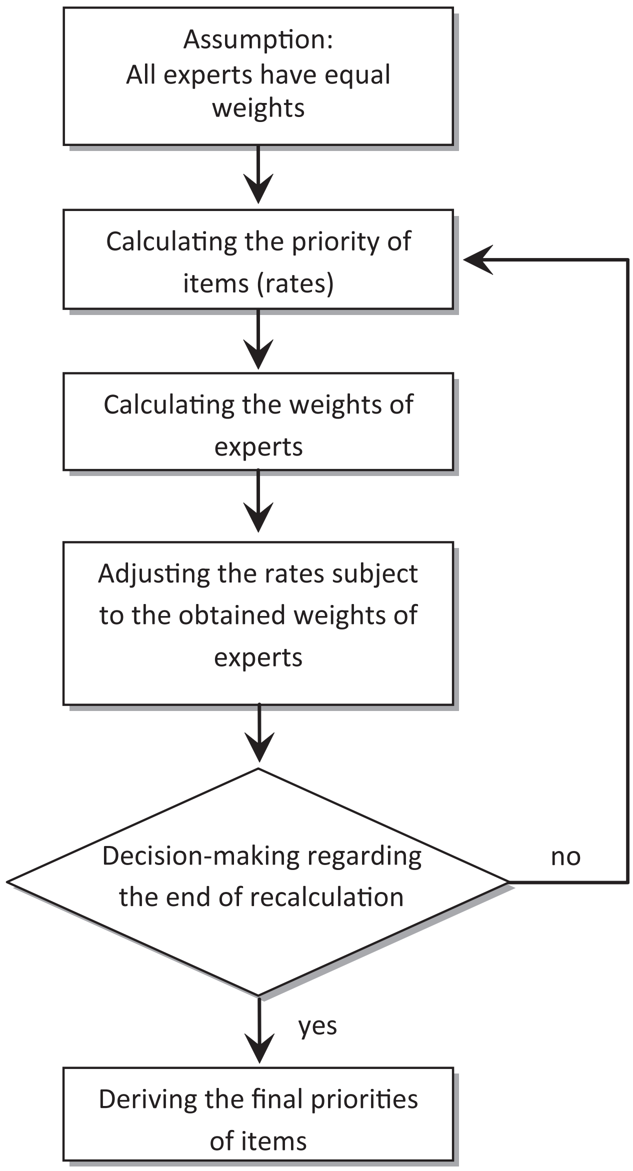


Fig. 3. Final scale construction algorithm

If the expert group is sufficiently large (more than 10 people), at the beginning of the procedure, the extreme values of item priority estimation can be discarded.

## Conclusion

The above item ranking procedure that includes the algorithm of final scale construction and takes into consideration the experts' competence, may prove to be useful in solving the problem of defining the preferability of items based on certain features, the problem of estimation of potential hazard and risk of items as part of analysis of structurally complex systems, etc.

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## The author's contribution

The author suggested a method of using the transitive graph of a Markovian process as part of expert ranking of items of a certain parent entity. It is proposed to use steady-state probabilities of a Markovian process as the correlation of priorities (weights) of the compared items. The paper sets forth an algorithm for constructing the final scale of comparison taking into consideration the experts' level of competence.

## Conflict of interests

The author declares the absence of a conflict of interests