

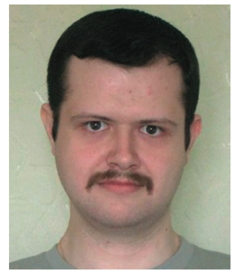

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# THE MATHEMATICAL MODEL OF COMPUTER NETWORKS' RELIABILITY

*The paper has investigated mathematical models of the reliability of LAN with an arbitrary topology and recoverable components, with the analysis allowing to obtain the ratios of probabilities of data loss in a network, a failure probability and expectation of time to failure of a network's components, without any assumptions about the laws of distribution of random variables. It has also offered the model for accounting multiple failures of network components.*

***Keywords:*** *distribution function, expectation, process of recovery, loss of data, multiple failures, server, client, port, period of regeneration.*

## 1. Introduction

Nowadays, computer technology is increasingly being used in human activity. Further industrialization of production and, consequently further enhancement of efficiency in all spheres of social activity is impossible without the use of information and communication technologies. Along with the rapid development and the enhancement of information processes, a number of problems emerge, and if we do not solve them, we cannot even talk about the efficiency of informatization. These issues should include the creation of computer systems (networks) described by sufficiently high reliability of their operation, and data protection contained in information systems from unauthorized access.

Most information systems (IS) are constructed based on computer technology and are a combination of software and hardware. Their contribution to the system unreliability as a whole has increased with the complexity of software, and it should be taken into account, while previously only the analysis of hardware reliability has been carried out. However, for the consumer no matter what was the cause of a system failure, the fact of a failure is important. Existing reliability models do not allow considering the hardware implementation and software together as a comprehensive whole, making them difficult to use because of the large diversity of applied systems, software and the complexity of their structure.

In what follows, we shall understand IS as a set of hardware and software implementing three basic functions: storage, processing, and data protection, which allow presenting any information system in the form of three subsystems: data storage system (DSS), data processing system (DPS) and security system (SS). This IS partition is purely logical and does not reflect a specific implementation of the system, but in any real system it allows distinguishing a set of hardware and software modules that provide execution performance of each of the mentioned above functions. Abstracting from the actual hardware and considering logical subsystems provides obtaining a method of reliability calculation that can be applied to any LAN regardless of the hardware configuration, the scale of the system and its functions. We shall consider LAN as a set of hardware and software that implements the basic functions and includes the following:

*A server* as a component of computer system (CS) containing a data storage system (DSS) with its data transmission systems (DTS) and a security system (SS),

*A client* as a component of CS, including a data processing system (DPS) with its own DTS and security system (SS),

A *hub* serves to connect the client and server, which consists of DPS and SS.

A data storage system includes hardware and software that provide functions of receiving, storing and releasing of information. The functions of a data processing system are the conversion of data and communications of DSS with the user. A security system executes functions of control over the work of other systems and data integrity retention. It must either prevent the loss of information or signal the impossibility of data protection. Here we assume a loss of data as LAN accident. We understand the loss of data as its actual destruction, or the inability to get access to it for a sufficiently long time.

Our objective is to develop a mathematical model of reliability of any LAN with recoverable components, taking into account the consequences of failure of subsystems and obtaining appropriate indices of reliability, as well as accounting of multiple failures of network components. This work is a further development of studies presented in [1, 2], where the formulas for calculating the probability of networks' data loss have been obtained. Evaluation of reliability indices and efficiency of redundant network structures with the selection of two groups of nodes and communication subsystem has been considered in [3].

## 2. Solution of the problem

First, we shall introduce the necessary notation. Let the random variable $\chi$ denote the DPS, time to failure, and the random variable $\gamma$ is the DPS, recovery time. Next, we shall denote times to failure of a client's DTS, port hub and server as random variables $\phi_1$, $\phi_2$ and $\phi_3$, and the values $\Psi_1$, $\Psi_2$ and $\Psi_3$ will present recovery time after failure of a client DTS, port hub and server respectively. We shall denote random time to failure of DSS by $\omega$, and the time of its recovery – by $\varepsilon$. Further the random variables $\xi_1$, $\xi_2$ and $\xi_3$ will present operating time to latent failures of a client's SS, port hub and server respectively. We shall denote the recovery time of a client's SS, port hub and server by random variables $\eta_1$, $\eta_2$ and $\eta_3$.

Since we assume that the system under consideration consists of recoverable subsystems and failures are detected during preventive maintenance, then we should include periods of preventive maintenance of a client's SS $T_1$, port hub $T_2$ and server $T_3$, to the consideration, and the duration of preventive maintenance of a client's SS, port hub and server we shall denote by $\theta_1$, $\theta_2$ and $\theta_3$, respectively, and the security system cannot perform its function during the control preventive maintenance. We shall assume that all distribution functions introduced in consideration of random variables are continuous, i.e. they possess density functions and all random variables are mutually independent. Let us consider the reliability model of the network components separately.

## 3. Mathematical model of the functioning of the client process reliability

We shall consider in more detail the process of client operation, which includes a data processing system (DPS) with its own DTS and SS, and at that, the client DPS and DTS are connected in series. Because of such a connection any undetected failure of these systems leads to the client failure and its further recovery, therefore the operation process of the client has regenerative times. Let the client operate in such a way that in the course of its operation, there are two alternatives of events' development.

The first alternative: DPS may fail earlier than DTS and if SS is in good order at this point of time, then the client's stop will occur, and DPS recovery will take place, and at that, we assume that at the end of DPS recovery DTS will also be fully restored, otherwise, if the SS at this point of time is in down state, then the client failure occurs.

The second alternative: DTS may fail earlier than DPS and if SS is in good order at this point of time, then the client's stop will occur, and the subsequent recovery of DTS will take place, and after the end of DTS recovery, DPS will also be fully restored, or if the SS at this point of time is in down state, then the client failure occurs. With this assumptions the *i*-th cycle of regeneration of the client operation process has a duration equal to the following

$$(\chi_i+\gamma_i)J_{\chi_i\le\phi_{1i}}+(\phi_{1i}+\Psi_{1i})J_{\phi_{1i}<\chi_i}\text{, where } J_{w\in A}=\begin{cases}1, & w\in A,\\ 0, & w\notin A.\end{cases}.$$

Consequently, if DPS failed earlier than DTS, then the regeneration cycle will consist of DPS time to failure and the client recovery time after the failure of DPS, and if the DTS failed earlier than DPS, then the regeneration cycle consists of DTS time to failure and the client recovery time after DTS failure. The cycle of regeneration, with the number *k* in which client failure will occur, has the duration equal to $\chi_\kappa J_{\chi_\kappa\le\phi_{1\kappa}}+\phi_{1\kappa}J_{\phi_{1\kappa}<\chi_\kappa}$. Thus, client's time to accident will consist of the total of $\kappa-1$ complete cycles of regeneration and the regeneration cycle, on which the failure occurred [3,5]. Then the distribution function of the client time operation to the first failure $\rho$ can be written as:

$$F_\rho(t)=P(\rho\le t)=P\left(\sum_{i=1}^{\kappa-1}\left((\chi_i+\gamma_i)J_{\chi_i\le\phi_{1i}}+(\phi_{1i}+\Psi_{1i})J_{\phi_{1i}<\chi_i}\right)+\chi_\kappa J_{\chi_\kappa\le\phi_{1\kappa}}+\phi_{1\kappa}J_{\phi_{1\kappa}<\chi_\kappa}\le t\right)=$$
$$=P\left(\sum_{i=1}^{\kappa}(\chi_i\wedge\phi_{1i})+\sum_{i=1}^{\kappa-1}(\gamma_iJ_{\chi_i\le\phi_{1i}}+\Psi_{1i}J_{\phi_{1i}<\chi_i})\le t\right),$$

where *k* is the number of the cycle in which the client failure occurred. Using the formula of total probability, we will rewrite the function $F_\rho(t)$ in the following form:

$$F_\rho(t)=\sum_{n=1}^{\infty}P\left(\sum_{i=1}^{n}(\chi_i\wedge\phi_{1i})+\sum_{i=1}^{n-1}(\gamma_iJ_{\chi_i\le\phi_{1i}}+\Psi_{1i}J_{\phi_{1i}<\chi_i})\le t\,\big|_{\kappa=n}\right)P(\kappa=n),$$

where $r$ is the probability of the client failure during the regeneration cycle. Taking into account that the probability of the failure occurred on the *n*-th cycle is equal to $P(\kappa=n)=r_1(1-r_1)^{n-1}$, then we have the following expression:

$$F_\rho(t)=\sum_{n=1}^{\infty}P\left(\sum_{i=1}^{n}(\chi_i\wedge\phi_{1i})+\sum_{i=1}^{n-1}(\gamma_iJ_{\chi_i\le\phi_{1i}}+\Psi_{1i}J_{\phi_{1i}<\chi_i})\le t\,\big|_{\kappa=n}\right)r_1(1-r_1)^{n-1}, \qquad (1)$$

Now we shall find $r_1$. First, we should introduce two additional random variables $U_n$ and $V_n$, taking into account the periodic checks of hardware. These values having a sense: $U_n$ is operation time of the

SS's client to the $n$-th failure, $V_n$ is operation time of the SS's client to the $n$-th recovery, defined by the following relations [4]:

$$\sum_{i=1}^{n}\xi_{1i}+\sum_{i=1}^{n-1}\left((T+\theta)-\left\{\frac{\xi_{1i}}{T+\theta}\right\}(T+\theta)\right)+\sum_{i=1}^{n-1}\eta_{1i}=U_n,$$

$$\sum_{i=1}^{n}\xi_{1i}+\sum_{i=1}^{n}\left((T+\theta)-\left\{\frac{\xi_{1i}}{T+\theta}\right\}(T+\theta)\right)+\sum_{i=1}^{n}\eta_{1i}=V_n,$$

where $[x]$ is the integer part of $x$, $\{x\}$ is the fractional part of $x$, $\chi\wedge\phi_1=\min(\chi,\phi_1)$.

Obviously, if SPD or DPS failure occur during the SS's recovery, then the client failure will also occur. The condition of a failure in this case is written down as $U_n\le\chi\wedge\phi_1<V_n$. However, the client failures can also occur when the failure of DTS or DPS fall on the time of a control preventive maintenance. The condition of failure can be rewritten down as sequence of the following events:

$$V_{n-1}+T_1\le\chi\wedge\phi_1<V_{n-1}+(T_1+\theta_1);\quad V_{n-1}+(T_1+\theta_1)+T_1\le\chi\wedge\phi_1<V_{n-1}+2(T_1+\theta_1);\ldots$$
$$V_{n-1}+\left(\left[\xi_{n1}/(T_1+\theta_1)\right]-1\right)(T_1+\theta_1)+T_1\le\chi\wedge\phi_1<V_{n-1}+\left[\xi_{n1}/(T_1+\theta_1)\right](T_1+\theta_1).$$

In view of a failure condition, the probability of the failure during the time of regeneration is defined as follows:

$$r_1=\sum_{n=1}^{\infty}\int_0^{\infty}(P(U_n\le x)-P(V_n\le x)+$$
$$+\,M\sum_{i=1}^{\left[\xi_{n1}/(T_1+\theta_1)\right]}(P(V_{n-1}+(i-1)(T_1+\theta_1)+T_1\le x)-P(V_{n-1}+i(T_1+\theta_1)\le x))dF_{\chi\wedge\phi_1}(x).$$

Omitting simple transformations, we have the following according to [4, 5]:

$$r_1\approx\frac{M\eta_1+M(T_1+\theta_1+\left[\xi_1/(T_1+\theta_1)\right](T_1+\theta_1))-M\xi_1}{M\eta_1+M(T_1+\theta_1+\left[\xi_1/(T_1+\theta_1)\right](T_1+\theta_1))}+\frac{M\left[\xi_1/(T_1+\theta_1)\right]\theta_1}{M\eta_1+M(T_1+\theta_1\left[\xi_1/(T_1+\theta_1)\right](T_1+\theta_1))}=$$
$$=1-\frac{M\xi_1-M\left[\xi_1/(T_1+\theta_1)\right]\theta_1}{M\eta_1+T_1+\theta_1+M\left[\xi_1/(T_1+\theta_1)\right](T_1+\theta_1)}=1-K_{г}^{СБ}=K_{нг}^{СБ},$$

where $K_{г}^{СБ}$ is the stationary availability of SS, $K_{нг}^{СБ}=1-K_{г}^{СБ}$ [4].

Now we shall rewrite the formula (1) in the form of convolutions of distribution functions:

$$F_{\rho}(t)=\sum_{n=1}^{\infty}(F_{\chi\wedge\phi_1}^{*(n)} * F_{\sigma}^{*(n-1)})(t)K_{нг}^{СБ}(1-K_{нг}^{СБ})^{n-1},$$

where $F_{\chi\wedge\phi_1}(t)=P(\chi\wedge\phi_1\le t)$, $F_{\acute{o}}(t)=P\left(\gamma\, J_{\chi\le\phi_1}+\Psi_1 J_{\phi_1<\chi}\le t\right)$. Then we shall represent the function $F_{\rho}(t)$ as follows:

$$\tilde{F}_{\rho}(s)=\sum_{n=1}^{\infty}(\tilde{F}_{\sigma}(s))^{n-1}(\tilde{F}_{\chi\wedge\phi_1}(s))^{n}K_{нг}^{СБ}(1-K_{нг}^{СБ})^{n-1}=\frac{K_{нг}^{СБ}\tilde{F}_{\chi\wedge\phi_1}(s)}{1-(1-K_{нг}^{СБ})\tilde{F}_{\sigma}(s)\tilde{F}_{\chi\wedge\phi_1}(s)},\tag{2}$$

where $\tilde{F}_{\alpha}(s)=\int_0^{\infty}e^{-st}dF_{\alpha}(t)=Me^{-s\alpha}$ is the Laplace-Stieltjes transform. Using the property of the Laplace-Stieltjes transform, we determine the mean time to the client's first failure under the formula, which we shall write after transform as follows:

$$M\rho=M(\chi\wedge\phi_1)+\frac{1-K_{нг}^{СБ}}{K_{нг}^{СБ}}(M(\chi\wedge\phi_1)+M\sigma),\tag{3}$$

where $M(\chi\wedge\phi_1)=\int_0^{\infty}\int_0^{y}x dF_{\chi}(x)dF_{\phi_1}(y)+\int_0^{\infty}\int_0^{y}x dF_{\phi_1}(x)dF_{\chi}(y)$ and $M\sigma=M\gamma\, MF_{\chi}(\phi_1)+M\Psi_1\overline{F}_{\chi}(\phi_1)$.

Distribution functions $F_{\chi\wedge\phi_1}(t)$ and $F_{\sigma}(t)$ can be written down in explicit form

$$F_{\sigma}(t)=F_{\gamma}(t)\int_0^{\infty}F_{\chi}(x)dF_{\phi_1}(x)+F_{\Psi_1}(t)\int_0^{\infty}F_{\phi_1}(x)dF_{\chi}(x),\; F_{\chi\wedge\phi_1}(t)=F_{\chi}(t)+F_{\phi_1}(t)-F_{\chi}(t)F_{\phi_1}(t).$$

## 4. A mathematical model of the hub port reliability

Let us calculate the distribution function of time to the first failure of a port hub. The hub serves to connect the clients and the server, and consists of DTS and SS. In working out the port hub mathematical model of reliability, we shall consider the overlay of regenerative processes of the data transmission system's operation and security system. The DTS's operation process in this context is crucial because it specifies moments of the regeneration process of the port operation, which is described by duration of the regeneration cycle $\phi_2+\Psi_2$. Distribution function of time to the first failure of the port $\rho'$ may be written as follows:

$$P\{\rho'\le t\}=P\left\{\sum_{k=1}^{\kappa-1}(\phi_{2k}+\Psi_{2k})+\phi_{2\kappa}\le t\right\},$$

where $k$ is the regeneration cycle, on which the port failure has occurred.

Since the process of the port functioning is an alternating renewal process, the probability that the failure has occurred at the $n$-th period of regeneration can be expressed as $P\{\kappa = n\} = r_2(1-r_2)^{n-1}$, where $r_2$ is the probability of an failure on the cycle of regeneration. Using the formula of total probability, we shall write

$$P\{\rho' \le t\} = \sum_{n=1}^{\infty} P\left\{\sum_{k=1}^{n-1}(\phi_{2k} + \Psi_{2k}) + \phi_{2n} \le t\right\} P\{\kappa = n\}.$$

Substituting the probability $P\{\nu = n\}$ in the recorded ratio, we obtain the desired probability in the following form:

$$F_{\rho'}(t) = r_2 \sum_{n=1}^{\infty} F_{\sum_{k=1}^{n-1}(\phi_{2k}+\Psi_{2k})+\phi_{2n}}(t)(1-r_2)^{n-1}.$$

Since the function $F_{\sum_{k=1}^{n-1}(\phi_{2k}+\Psi_{2k})+\phi_{2n}}(t)$ is the convolution of the distribution functions $F_{\phi_2}^{*(n)}(t)$ and $F_{\Psi_2}^{*(n)}(t)$, then we shall rewrite $F_{\rho'}(t)$ as follows:

$$F_{\rho'}(t) = r_2 \sum_{n=1}^{\infty} (F_{\phi_2}^{*(n)} * F_{\Psi_2}^{*(n-1)})(t)(1-r_2)^{n-1}. \tag{4}$$

Carrying out the Laplace-Stieltjes transform for the time distribution function before the first failure of the hub port, we shall rewrite (4) as follows:

$$\tilde{F}_{\rho'}(s) = q_2 \sum_{n=1}^{\infty} (\tilde{F}_{\phi_2}(s))^n (\tilde{F}_{\Psi_2}(s))^{n-1}(1-q_2)^{n-1} = \frac{q_2 \tilde{F}_{\phi_2}(s)}{1-(1-q_2)\tilde{F}_{\phi_2}(s)\tilde{F}_{\Psi_2}(s)}, \tag{5}$$

where $\tilde{F}_{\phi_2}(s)$, $\tilde{F}_{\Psi_2}(s)$, $\tilde{F}_{\rho'}(s)$ are the Laplace-Stieltjes transforms of the distribution functions $F_{\phi_2}(t)$, $F_{\Psi_2}(t)$ and $F_{\rho'}(t)$.

Thus, it has been possible to obtain the Laplace-Stieltjes transform of the distribution function of time to the first failure occurrence of the complex in the easy-to-compute form of the mathematical time expectation to the first failure of the hub port. Since $M\rho' = -\frac{d\tilde{F}_{\rho'}(s)}{ds}\Big|_{s=0}$, then

$$M\rho' = M\phi_2 + \frac{1-r_2}{r_2}(M\phi_2 + M\Psi_2), \tag{6}$$

where $M\phi_2$ and $M\Psi_2$ are expectations of the random variables $\phi_2$ and $\Psi_2$.

As it has been shown previously, the probability of a hub port failure at regeneration cycle $r_2$ for the client can be calculated by the following ratio:

$$r_2 \approx K_{2í\, ã}^{ÑÁ} = 1 - \frac{M\xi_2 - M\left[\xi_2 \big/ (T_2+\theta_2)\right]\theta_2}{M\eta_2 + M\left((T_2+\theta_2) + \left[\xi_2 \big/ (T_2+\theta_2)\right](T_2+\theta_2)\right)}.$$

## 5. The mathematical model of server reliability

The server includes a data storage system (DSS) with its own data transmission system (DTS) and security system (SS). The development of the process of a server operation can occur according to the two scenarios: if DSS failed earlier than DTS, then the regeneration cycle consists of the sum of DSS time to failure and server recovery time after DSS failure, and if the DTS failed earlier than DSS, the regeneration cycle consists of the sum of DTS time to failure and server recovery time after DTS failure. With this assumption, the *i*-th cycle of the process regeneration of the server operation has duration $\tau_i$ equal to $\omega_i \wedge \phi_{3i} + \varepsilon_i J_{\omega_i \le \phi_{3i}} + \Psi_{3i} J_{\phi_{3i} < \omega_i}$. The regeneration cycle, with the number *k* during which the server failure occurs will not be complete, and therefore it has duration equal to $\omega_\kappa J_{\omega_\kappa \le \phi_{3\kappa}} + \phi_{3\kappa} J_{\phi_{3\kappa} < \omega_\kappa} = \omega_\kappa \wedge \phi_{3\kappa}$. Server's time to failure is equal to the sum of $\kappa - 1$ complete cycles of regeneration and the regeneration cycle on which the failure occurred.

Because server reliability indices will be calculated in the same way that the reliability indices of the client, the presentation of this model will be carried out without further details. So, the distribution function of server operating time to the first failure $\rho''$ can be written down as follows:

$$F_{\rho''}(t) = \sum_{n=1}^{\infty} P\left(\sum_{i=1}^{n}(\omega_i \wedge \phi_{3i}) + \sum_{i=1}^{n-1}(\varepsilon_i J_{\omega_i \le \phi_{3i}} + \Psi_{3i} J_{\phi_{3i} < \chi_i}) \le t\right) r_3 (1 - r_3)^{n-1}, \quad (7)$$

where $r_3$ is the probability of a server failure during the regeneration cycle, and the probability that the server failure occurred just on the *n*-th cycle is equal $P(\kappa = n) = r_3(1-r_3)^{n-1}$. Without dwelling on discussion of methods to obtain a ratio of the probability of the server failure during the regeneration cycle, we shall immediately write down the final ratio, as these methods are described in detail earlier. So $r_3$ has the following form:

$$r_3 = 1 - \frac{M\xi_3 - M\left[\xi_3 \big/ (T_3+\theta_3)\right]\theta_3}{M\eta_3 + M(T_3 + \theta_3 + \left[\xi_3 \big/ (T_3+\theta_3)\right](T_3+\theta_3))} = 1 - K_{3г}^{СБ} = K_{3нг}^{СБ},$$

where $K_{3г}^{СБ}$ is the stationary availability factor of the server's SS.

The probability (7) is calculated from the sum of random variables, so it should written down in the form of a convolution of the two probabilities of functions $F_{\omega \wedge \phi_3}(t) = P(\omega \wedge \phi_3 \le t)$, $F_{\sigma_3}(t) = P(\varepsilon J_{\omega \le \phi_3} + \Psi_3 J_{\phi_3 < \omega} \le t)$, then we have the following:

$$F_{\rho''}(t) = \sum_{n=1}^{\infty} (F_{\omega \wedge \phi_3}^{*(n)} * F_{\sigma_3}^{*(n-1)})(t) K_{3нг}^{СБ} (1 - K_{3нг}^{СБ})^{n-1}. \quad (8)$$

By carrying out the Laplace-Stieltjes transform for the ratio (8), we obtain:

$$\tilde{F}_{\rho''}(s)=\frac{K_{3нг}^{СБ}\tilde{F}_{\omega\wedge\phi_3}(s)}{1-(1-K_{3нг}^{СБ})\tilde{F}_{\omega\wedge\phi_3}(s)\tilde{F}_{\sigma_3}(s)}. \quad (9)$$

In finding the mean median time to first server's failure $M\rho''$ we shall use the standard method, namely $M\rho''=-\frac{d\tilde{F}_{\rho''}(s)}{ds}\Big|_{s=0}$ and after the appropriate calculations, we obtain the following:

$$M\rho''=M(\omega\wedge\phi_3)+\frac{1-K_{нг}^{СБ}}{K_{нг}^{СБ}}(M(\omega\wedge\phi_3)+M\sigma_3) \quad (10)$$

where $M(\omega\wedge\phi_3)=\int_0^{\infty}\int_0^{y}xdF_{\omega}(x)dF_{\phi_3}(y)+\int_0^{\infty}\int_0^{y}xdF_{\phi_3}(x)dF_{\omega}(y)$ and $M\sigma_3=M\varepsilon\, MF_{\omega}(\phi_3)+M\Psi_3 M\bar{F}_{\omega}(\phi_3)$

The distribution functions $F_{\chi\wedge\phi_1}(t)$ and $F_{\sigma_3}(t)$ can be written explicitly

$$F_{\sigma_3}(t)=F_{\varepsilon}(t)\int_0^{\infty}F_{\omega}(x)dF_{\phi_3}(x)+F_{\psi_3}(t)\int_0^{\infty}F_{\phi_3}(x)dF_{\omega}(x),\ F_{\omega\wedge\phi_3}(t)=F_{\omega}(t)+F_{\phi_3}(t)-F_{\omega}(t)F_{\phi_3}(t).$$

Note that the ratios (2), (5) and (9), for example, for the exponential distributions are rational algebraic functions, their original is determined sufficiently easily with the standard methods.

## 6. Mathematical model of network reliability

Let us suppose that a network consists of a finite number of components *N*. The function of the network reliability *S* shall be denoted by $h=h_S(p_1,p_2,...,p_N)=M\phi_S(x)$, where $\phi(x)=\phi(x_1,x_2,...,x_N)$ is the structure function (boolean function) [6], and $P_i=P(x_i=1)=Mx_i$ is the probability of failure-free operation of *i*-th component of the network at time *t*.

Because the network is a recoverable system, then the whole time $[0,\infty)$ of a component operation splits into separate cycles, in each of which the component operates during part-time without accidents (the set of time intervals $Q^+$), and the rest of the time is spent on an failure consequences' elimination (the set of time intervals $Q^-$). We shall call the function of sets $A_1$, $A_2$, $A_3,\ldots,A_N$ $\widehat{\phi}_S(A)=\widehat{\phi}_S(A_1,A_2,...,A_N)$ as dual function for the Boolean function $\phi_S(x)=\phi_S(x_1,x_2,...,x_N)$. The function of sets is determined according to the following rule: if $\phi_S(x_1,x_2)=x_1\wedge x_2$, then $\widehat{\phi}_S(A_1,A_2)=A_1\cap A_2$, and for $\phi_S(x_1,x_2)=x_1\vee x_2$, $\widehat{\phi}_S(A_1,A_2)=A_1\cup A_2$, finally for $\phi_S(x)=\overline{x}\ \Rightarrow\ \widehat{\phi}_S(A)=(0,\infty)-A$.. Formally, the function $\widehat{\phi}_S(A_1,A_2,...,A_N)$ can be introduced by using the complete disjunctive normal form (CDNF) of the function $\widehat{\phi}_S(A_1,A_2,...,A_N)$. If CDNF of the function $\widehat{\phi}_S(A_1,A_2,...,A_N)$ is given in the form:

$\phi_S(x_1,x_2,...,x_N)=\sum_{\alpha}C_{\alpha_1,\alpha_2,...,\alpha_N}x_1^{\alpha_1}x_2^{\alpha_2}...x_N^{\alpha_N}$, then $\widehat{\phi}_S(A_1,A_2,...,A_N)=\bigcup_{\alpha}C_{\alpha_1,\alpha_2,...,\alpha_N}A_1^{\alpha_1}A_2^{\alpha_2}\ldots A_N^{\alpha_N}$,
where

$$A^{\alpha} = \begin{cases} A, & \text{if } \alpha = 1 \\ (0,\infty) - A, & \text{if } \alpha = 0, \end{cases} \qquad C_{\alpha_1,\alpha_2,\ldots,\alpha_N} B = \begin{cases} 0, & \text{if } C_{\alpha_1,\alpha_2,\ldots,\alpha_N} = 0 \\ B, & \text{if } C_{\alpha_1,\alpha_2,\ldots,\alpha_N} = 0 \end{cases}$$

The mapping $\pi\ \phi \to \widehat{\phi}$ is a natural morphism of the category of Boolean functions in the category of sets of functions; the meaningful sense is that if $A_i$ is a set of those times at which the component $x_i$ is in good order, then $\widehat{\phi}_S(A_1, A_2, ..., A_N)$ is a set of moments of the system correct operation.

Let $t$ be some point of time, then, by definition, the structure function $\widehat{\phi}_S(Q_1^+ \cap [0,t], Q_2^+ \cap [0,t], Q_3^+ \cap [0,t], ..., Q_n^+ \cap [0,t])$ is a set of points of time, at which the system was in good condition until the point of time $t$. Then the average time for the correct operation of the system during time $t$ will be determined by the following formula $K_S(t) = Mmes\widehat{\phi}_S(Q_1^+ \cap [0,t], Q_2^+ \cap [0,t], Q_3^+ \cap [0,t], ..., Q_n^+ \cap [0,t])$, where $mesA$ is the Lebesgue measure of the set $A$. Let us transform $K_S(t)$ as follows

$$K_S(t) = M\int_0^{\infty} J_{\widehat{\phi}_S(Q_1^+ \cap [0,t], Q_2^+ \cap [0,t], Q_3^+ \cap [0,t], ..., Q_n^+ \cap [0,t])}(y)dy.$$

Taking into account the sense of events' index and that if for $y > t$ the following $y \notin Q_i^+ \cap [0,t]$ holds true, we have

$$K_s(t) = \int_0^{\infty} P\left\{y \in \widehat{\phi}_S(Q_1^+ \cap [0,t], Q_2^+ \cap [0,t], Q_3^+ \cap [0,t], ..., Q_n^+ \cap [0,t])\right\}dy = \int_0^{t} h_s(P_1(y), P_2(y), ..., P_n(y))dy. \quad (11)$$

It is necessary to remember that $P_i(t) = P(\rho_t = 1)$ is the probability of failure-free operation of the $i$-th component at time point $t$. Thus, the ratio (11) allows reducing the computation of $K_S(t)$ to the calculation of $P_i(t) = P(\rho_t = 1)$ for individual components. Let the time to failure $\rho_i$ of the $i$-th component be a random variable. After the failure, the component is recovered over time $\delta_i$, which is also a random variable. It is assumed that the recovery is complete, i.e. times of correct operation of the component after its recovery, as well as after the second and subsequent recoveries are independent and have the same distribution as random variables $\rho_i$, $\delta_i$. We shall denote the joint distribution of random variables $\rho_i$, $\delta_i$ by $F_i(A,B) = P(\rho_i \in A, \delta_i \in B)$ and in case of the independence of these variables $F_i(A,B) = P(\delta_i \in B)P(\rho_i \in A) = F_{\rho_i}(A)G_{\delta_i}(B)$.

When calculating the probability of accident-free operation of the $i$-th component, we will use the total probability formula, according to which we have

$$P_i(t) = P(t \in Q_i^+) = \int_0^{\infty}\int_0^{\infty} P(t \in Q_i^+ \mid \rho_{i1} = x, \delta_{i1} = y)dF_{\rho_i}(x)dG_{\delta_i}(y),$$

where $\rho_{i1}$ and $\delta_{i1}$ are random times to the first failure and to the first recovery of the component respectively. As the process of the $i$-th component operation is an alternating recovery process, then its operation time is covered by disjoint cycles of regeneration with duration $\rho_i + \delta_i = \tau_i(\rho_i, \delta_i)$. There are two options of the process operation development: $\tau_i(\rho_i, \delta_i) > t$ and $\tau_i(\rho_i, \delta_i) \le t$. Then the probability $P_i(t)$ of an accident-free operation of the $i$-th component will be calculated as

$$P_i(t) = \iint\limits_{\tau_i(x,y)\le t} P\left(t \in Q_i^+ \big|_{\rho_{i1}=x,\delta_{i1}=y}\right) dF_{\rho_i}(x) dG_{\delta_i}(y) + \iint\limits_{\tau_i(x,y)>t} P\left(t \in Q_i^+ \big|_{\rho_{i1}=x,\delta_{i1}=y}\right) dF_{\rho_i}(x) dG_{\delta_i}(y) = I_1 + I_2.$$

The condition $\tau_i(\rho_i, \delta_i) > t$ means that the moment of regeneration of the *i*-th component come after time *t,* and, consequently, the condition $t \in Q_i^+$ is equivalent to the condition $t \in [0, \rho_{i1}]$ and

$$I_2 = \iint\limits_{x+y>t} P\left(t \in [0,\rho_{i1}] \big|_{\rho_{i1}=x,\delta_{i1}=y}\right) dF_{\rho_i}(x) dG_{\delta_i}(y) = \iint\limits_{x+y>t} \left(J_{t\in[0,x]}\right) dF_{\rho_i}(x) dG_{\delta_i}(y) = 1 - F_{\rho_i}(t).$$

Considering the option, $\rho_i + \delta_i \le t$, it should be taken into account that time point $\rho_{i1} + \delta_{i1}$ is the moment of the first regeneration of the component, i.e. at this time point, the operability of the component is fully recovered, and after that, the component continues correct operation during some more time $t - \delta_{i1} - \rho_{i1}$. In view of these considerations, we can rewrite $I_1$ in the following form:

$$I_1 = \iint\limits_{\tau_i(x,y)\le t} P\left(t \in [0,\rho_{i1}] \big|_{\rho_{i1}=x,\delta_{i1}=y}\right) dF_{\rho_i}(x) dG_{\delta_i}(y) = \iint\limits_{\tau_i(x,y)\le t} P_i(t-x-y) dF_{\rho_i}(x) dG_{\delta_i}(y) = \int_0^t P_i(t-z) dF_{\tau_i}(z).$$

By summing up $I_1$ and $I_2$, we obtain the integral equation in convolutions

$$P_i(t) = g_i(t) + \int_0^t P_i(t-z) dF_{\tau_i}(z), \tag{12}$$

where $g_i(t) = \bar{F}_{\rho_i}(t)$, $F_{\tau_i}(t) = P\left(\rho_i + \delta_i \le t\right)$.

In terms of the Laplace-Stieltjes transform, the solution (12) has the following form:

$$\tilde{P}_i(s) = \tilde{g}_i(s) + \tilde{P}_i(s)\tilde{F}_{\tau_i}(s), \text{ or } \tilde{P}_i(s) = \frac{\tilde{g}_i(s)}{1-\tilde{F}_\tau(s)} = \frac{1-Me^{-s\rho_i}}{1-Me^{-s(\rho_i+\delta_i)}}.$$

As mentioned above, it is not possible to convert the obtained solution in general form, so here we write only its asymptotic value at $s \to 0$. Then $P_i = \lim_{s\to 0} \tilde{P}_i(s) = \frac{M\rho_i}{M\rho_i + M\delta_i} = K_{ei}$. Thus, the asymptotic ratio of $P_i(t)$ coincides with the component availability factor and, consequently, the asymptotic ratio for the probability of finding the network in failure state will be calculated as $P_a \cong 1 - h(K_{e1}, K_{e2}, ..., K_{eN})$. Let us give the ratio for the probability of the network stay in failure state of widely-spread LAN topologies: star, ring, common bus. Therefore, the network with the topology "common bus" transits into failure state, as soon as at least one of the clients has transited into this state. In this case the structure chart of the network is a serial connection of clients and a server, and the failure probability will be determined by the ratio $P_a = 1 - K_г^S \prod_{i=1}^{N} K_г^{K_i}$. The network having the topology "ring" transits into a failure state when at least one of the clients or the port by which connection is carried out, or the server has transited into this state. The failure probability for this topology can be written down as $P_a = 1 - K_г^S K_г^P \prod_{i=1}^{N} K_г^{K_i}$. Since the network with the topology "star" transits into failure state, if there was a loss of data on at least one of the clients, or hub port, or on the server, then the failure probability is equal to $P_a = 1 - K_г^S \prod_{i=1}^{N+1} K_г^{P_i} \prod_{i=1}^{N} K_г^{K_i}$, where $N$ is the number of clients in the network, $K_г^{K_i}$ is the availability factor of its client, $K_г^{P_i}$ is the availability factor of the $i$-th hub port, $K_г^S$ is the availability factor of server.

Thus, a mathematical model of reliability is presented above and expressions for calculating the probability of data loss (an accident) and mean time to the first failure of an arbitrary network with recoverable component are obtained. However, this does not take into account the possibility of data loss due to the network power supply failure. The network power supply failure leads to the so-called *multiple failures or common cause failures*.

To account a network component common cause failures, we shall introduce the following notation. Let the random variable $\zeta$ be the time to failure, and the random variable $\nu$ be the recovery time of the power supply system. The random variables $\zeta$ and $\nu$ are mutually independent variables with distribution functions $F_\zeta(t) = P(\zeta \le t)$ and $F_\nu(t) = P(\nu \le t)$. Let $\delta$ be the random time to failure of the redundant power supply system with distribution function $F_\delta(t) = P(\delta \le t)$. The correct functioning of the power supply system is critical for accident-free operation of the network, so we will assume that in the event of failure of the main power supply system, electricity supply can be carried out using the redundant system. During the operation of the redundant system, the main power supply system must be recovered, or a network failure will occur. We believe that the redundant system is able to maintain the functioning of the client only for a limited time. Obviously, if the main power supply system recovery is not completed before the redundant system failure, then the client will be cut off power, i.e. failure of the client will occur. At the same time, we shall assume that if the main power supply system will be recovered before the redundant system failure, then the redundant system immediately will return to its original state.

Thus, a network component failure may occur either on the internal reasons or due to a power supply loss. Let us introduce the notation $\rho$ as the time before the component failure due to internal reasons, $\beta$ as the time before the component failure due to the power supply loss and $\alpha$ as the time to the first failure of the component. Since the time to failure is the minimum of time to failure due internal reasons and time to failure due to loss of power supply, it is obvious that $\alpha = \rho J_{\beta > \rho} + \beta J_{\beta \le \rho} = \rho \wedge \beta$. Then the distribution function of the minimum of time to the first failure of a network component will be determined by the following ratio

$$F_\alpha(t) = P(\alpha \le t) = F_\rho(t) + F_\beta(t) - F_\rho(t) F_\beta(t). \tag{13}$$

Mean time to the first failure can be written down respectively as

$$M\alpha = M\rho + M\beta - \int_0^\infty (1 - F_\rho(t) F_\beta(t)) dt. \tag{14}$$

Now we determine the distribution of time to failure of a network component due to the failure of the system power supply. It is easy to notice that the distribution of time to failure due to power supply failure is defined as follows

$$P\{\beta \le t\} = P\left\{\sum_{k=1}^{\pi-1}(\zeta_k + \nu_k) + (\zeta_\nu + \delta_\nu) \le t\right\},$$

where $\pi$ is the cycle number of the regeneration process of power supply system operation, where the network component failure has occurred.

Taking into account that the process of a component operation is a regenerating process, and then it is easy to determine that $P\{\pi = n\} = q_{эл} p_{эл}^{n-1}$ is the probability that the failure occurred on the *n*-th cycle of regeneration, where $q_{эл}$ is the probability of failure on a cycle of regeneration, $p_{эл} = 1 - q_{эл}$. Then $P\{\beta \le t\}$ should be calculated according to the formula of total probability

$$F_\beta(t) = q_{эл}\sum_{n=1}^{\infty}(F_\zeta^{*(n)} * F_\nu^{*(n-1)} * F_\delta)(t)(1-q_{эл})^{n-1}$$

To calculate $P\{\beta \le t\}$ of the distribution function of time to the first failure of the s power supply system, we again will use the Laplace-Stieltjes transform and write down it in the following form:

$$\tilde{F}_\beta(s) = q_{эл}\sum_{n=1}^{\infty}(\tilde{F}_\zeta(s))^n (\tilde{F}_\nu(s))^{n-1}\tilde{F}_\delta(s)(1-q_{эл})^{n-1},$$

where $\tilde{F}_\zeta(s)$, $\tilde{F}_\nu(s)$ and $\tilde{F}_\delta(s)$ are the Laplace-Stieltjes transforms of corresponding distribution functions. The expectation of time before the first failure of a network component due to power supply failure can be written down as follows

$$M\beta = M\zeta + M\delta + \frac{1-q_{эл}}{q_{эл}}(M\zeta + M\nu) \tag{15}$$

Using the formula for the sum of a geometric progression, we obtain:

$$\tilde{F}_\beta(s) = \frac{q_{эл}\tilde{F}_\zeta(s)\tilde{F}_\delta(s)}{1-(1-q_{эл})\tilde{F}_\zeta(s)\tilde{F}_\nu(s)}. \tag{16}$$

Thus, it was possible to obtain the Laplace-Stieltjes transform of the distribution function of time before the first failure of a network component due to power supply failure. To use the ratio of the expectation of time before the first failure, it is necessary to calculate the probability of failure of a network component $q_{ýë}$ in the regeneration cycle of the process of power supply system operation due to failure of the power supply system. Obviously, the failure condition for *n*-th cycle can be presented by $\nu \ge \delta$. Then we shall write down the following expression

$$q_{эл} = P(\nu \geq \delta) = \int_0^\infty P(\delta \leq t) dF_\nu(t) = \int_0^\infty F_\delta(t) dF_\nu(t) = MF_\delta(\nu) .$$

The distribution function of time to failure of the first client can be determined by writing down the inverse transformation for (2) and (16) and by substituting the appropriate distribution function (13). The distribution function of time to the first accident of the port, we shall determine by writing down the inverse transformation for (5) and (16) and by substituting the corresponding distribution function in the ratio (13). The distribution function of time to the first accident of the server, we shall determine by writing down the inverse transformation for the ratio (9) and (16) and by substituting the corresponding distribution functions (13). The mean time to failure of the first client we shall determine from the ratios (3), (9) and (15), the mean time to failure of the port – from the ratios (5), (10) and (15), and the mean time to failure of the server – from the ratios (7), (10) and (15). Then, using the reliability function of the network, we shall write down the formulas for calculating the probability of a network failure based on probabilities of states of individual components.

## 7. The conclusion

Thus, the mathematical model of LAN's reliability of arbitrary structure with recoverable components has been developed. The expressions for calculating the probability of a network failure under the most general assumptions about the laws of distribution of random variables have been obtained. The ratios for the reliability determination of the probability of data loss in the LAN and the asymptotic ratios of the mean time to first failure of network components have been obtained also. Using these ratios, we can calculate the probability of finding a network with arbitrary topology in failure state due to multiple failures of components.

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