Plan of tests with addition. Efficient estimate of dependability indicators

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Abstract. *The* **Aim** *of the paper consists in improving the efficiency of dependability indicator estimation for the plan of tests with addition, i.e. probability of no-failure and mean time to failure. Due to economic considerations, determinative dependability tests of highly dependable and costly products involve minimal numbers of products, expecting failure-free testing or testing with one failure, thus minimizing the number of tested products. The latter case is most interesting. By selecting specific values of the acceptance number Q and number of tested products, the tester performs a preliminary estimation of the dependability indicator, while selecting Q = 1 the tester minimizes the risks caused by an unlikely random failure. However, as the value Q grows, the number of tested products does so as well, which makes the testing costly. Therefore, the reduction of the number of products tested for dependability is the first-priory problem and, in this context, economic planning of testing with addition is becoming increasingly important. We will consider binomial tests (original sample) with addition of one product (oversampling) to testing in case of failure of any of the initially submitted products. Testing ends when all submitted products have been tested with any outcome (original sampling and oversampling). Hereinafter it is understood that the testing time is identical for all products. Testing with the acceptance number of failures greater than zero (Q > 0) conducted with addition allows reducing the number of tested products through successful testing of the original sample.* **Methods.** *Efficient estimation is based on the integral approach formulated in many papers. The integral approach is based on the formulation of the rule of efficient estimate selection* $\hat{\theta}_0(\tau;n;k,m)$ *specified on the vertical sum of absolute (or relative) biases of estimates* $\hat{\theta}(n;k,m)$ *selected out of a certain set based on the distribution law parameter, where n is the number of products initially submitted to testing. The criterion of selection of an efficient estimate of the probability of failure (or PNF) at a set of estimates* $\hat{\theta}(\tau;n;k,m)$ *is based on the total square of absolute (or relative) biases of the mathematical expectation of estimates* $E\hat{\theta}(\tau;n;k,m)$ *from probability of failure p for all possible values of p, n.* **Conclusions.** *The paper examines the probability of no-failure estimates for the plan of tests with addition. For the case of n > 3, the estimates* $\hat{P}=1-\hat{p}=1-\frac{r}{n+k}$ *and composite estimate* $1-\overline{p}(\tilde{v}(\beta=0,5))$ *are more efficient in comparison with estimate* $1-\tilde{v}(\beta=0,5)$*. The composite estimate of the probability of no-failure* $1-\overline{p}(\tilde{v}(\beta=0,5))$ *should be used in failure-free tests. For the case of n > 3, testing with the acceptance number of failures greater than zero (Q > 0) conducted with addition allows reducing the number of tested products through successful testing of the original sample. The composite estimate of the mean time to failure* $\hat{T}_1=\frac{\tau}{-\ln(1-\overline{p}(k,m,n,\beta=0,6))}$ *is bias-efficient among the proposed mean time to failure estimates. The obtained composite estimates* $\overline{p}$ *and* $\hat{T}_1$ *are of practical significance in the context of failure-free testing with addition***.**

Keywords*: Bernoulli scheme, test plan, point estimate, probability of no-failure, efficient estimate, mean time to failure.*

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Introduction

Due to economic reasons, determinative dependability tests of highly dependable, costly products involve minimal numbers of products, expecting failure-free testing (acceptance number $Q = 0$) or testing with one failure ($Q = 1$), thus minimizing the number of tested products. The latter case is most interesting. By selecting specific values of the acceptance number Q and number of tested products, the tester performs a preliminary estimation of the dependability indicator, while selecting $Q = 1$ the tester minimizes the risks caused by an unlikely random failure. However, as the value Q grows, the number of tested products does so as well, which makes the testing costly. Therefore, the reduction of the number of products tested for dependability is the first-priory problem and, in this context, economic planning of testing with addition is becoming increasingly important [1].

Preparation of the plan of tests with addition

We will consider binomial tests (original sample) [1, 2] with addition of one product (oversampling) to testing in case of failure of any of the initially submitted products. Testing ends when all submitted products have been tested with any outcome (original sampling and oversampling). Hereinafter it is understood that the testing time is identical for all products.

Testing with the acceptance number of failures greater than zero ($Q > 0$) conducted with addition allows reducing the number of tested products through successful testing of the original sample.

The Aim of the paper

The aim of the paper consists in improving the efficiency of dependability indicator estimation for the plan of tests with addition, i.e. probability of no-failure (PNF) and mean time to failure (MTF).

Properties of probability of no-failure estimates for a plan of tests with addition

Let n be the number of tested same-type products initially submitted to testing, while $R=r$ is the number of failed products, including k failures out of n initially submitted products and m failures out of k subsequently submitted products, i.e. $r=k+m$. Then, the number of tested products will be $N=n+k$. For the sake of convenient formula writing, in some cases (where possible), the designations of random values will be identical to their representations. Let failures be independent events, then the probability of occurrence is equal to r failures over the testing period (hereinafter referred to as $P_n(R=r)$) will be expressed with the formula that results from the following procedure ($n \geq k \geq m; r = k + m \leq 2n$):

$$P_k(m) := C_k^m p^m q^{k-m};$$

$$P_n(k) := C_n^k p^k q^{n-k} \sum_{m=0}^{k} P_k(m) = C_n^k p^k q^{n-k},$$

where $q=1-p$, p is the probability of failure, C_n^k is the number of combinations k out of n elements.

$$P_n(k,m) := P_n(k) P_k(m) = C_n^k C_k^m p^{k+m} q^{n-m},$$

$$P_n(R = r) = \sum_{k=0}^{n} \sum_{m:m+k=r,\, m \leq k} P_n(k,m),$$

$$r = k + m = 0,1,2,\ldots,2n;\ k = 0,1,2,\ldots,n;$$
$$m : m + k = r, m \leq k.$$

Out of the definition of probability $P_n(k = x, m = y) = P_n(k = x) P_n(m = y)$, where $x, y = 0,1,2,\ldots,n$ and $P_n(R=r)$ we can easily obtain the probabilistic function of the plan of tests with addition:

$$P_{n\sum}(k \leq x, m \leq y) = \sum_{k=0}^{x} \sum_{m:m+k \leq x+y,\, m \leq k,\, m \leq y} P_n(k,m). \quad (1)$$

The average number of tested products over the period of testing with addition comprises the number of initially submitted products and the average number of those initially submitted products that failed, i.e. $N=n+np$. Then, the average number over the period of testing with addition will be $E(R,n)=Np=E(k,n)+E(m,n)=np+np*p=(n+np)p=n(p+p^2)$.

The PNF estimate $\hat{p} = \frac{r}{n+k}$ is efficient for a plan of tests with addition [1]. Let us examine the properties of the obtained estimate $\hat{p} = \frac{r}{n+k}$ and, as a consequence, PNF estimate $\hat{P} = 1 - \hat{p} = 1 - \frac{r}{n+k} = \frac{n-m}{n+k}$ [1].

Let $k+m=r>1$, $\hat{p} = \frac{r}{n+k} = \frac{r}{n+r-m}$, then for various $m_1>m_2$ the following inequality is fulfilled

$$\hat{p}(k_1 + m_1 = r; k_1, m_1) = \frac{r}{n+r-m_1} >$$
$$> \hat{p}(k_2 + m_2 = r; k_2, m_2) = \frac{r}{n+r-m_2}. \quad (2)$$

I.e. the dependability of the controlled batch of products subject to the results of testing of a sample, in which the number of failed products out of the initially submitted is higher, than in the sample of the compared batch of products under the same number of failures, will always be higher, than that of the compared batch of products. In other words, while comparing the results of two finalized samples (under the assumption of identical numbers of failures), the priority in terms of dependability is given to those products, whose failures primarily occurred within the initial sample, rather than the additional one. And in this regard oversampling enables remedial action in case of unsuccessful initial testing. That constitutes the advantage of the test plan with addition.

Unbiassed estimates

The mathematical expectation of the estimate $\hat{p}(n;k,m)=\dfrac{r}{n+k}$ will be expressed with formula [1]:

$$E\left(\hat{p}\left(n;k,m\right)\right)=\sum_{r=0}^{2*n}\frac{r}{n+k}P_n(r).$$

Estimate $\hat{p}(n;k,m)=\dfrac{r}{n+k}$, is generally biased $E\left(\hat{p}\left(n;k,m\right)\right)\neq p$ [1].

By equating mathematical expectation of the estimate $\hat{p}(n=1)$ to parameter p we can easily obtain the unbiassed estimate of the probability of failure $\hat{p}_1$ for the case of n=1 [1]:

$$\hat{p}_1=\begin{cases}0, r=0\\ 1, r>0\end{cases}=\begin{cases}0,\ r=0, k=0, m=0;\\ 1,\ r=1, k=1, m=0;\\ 1,\ r=2, k=1, m=1.\end{cases}$$

An unbiassed estimate is an indicator function, i.e. in case of failures estimate $\hat{p}_1$ becomes equal to one, otherwise to zero. The case of $n=1$ is practically uninteresting as it is the same as the binomial plan and thus is not further considered in this paper.

The mathematical expectation of the estimate $\hat{p}(n=2)=\dfrac{r}{2+k}$:

$$n=2:E\left(\hat{p}\right)=\sum_{r=0}^{4}\hat{p}(r)P\left(n=2,R=r\right).$$

The unbiassed estimate for parameter p in case $n=2$ will be expressed with formula [1]:

$$\hat{p}_2(k,m)\equiv\begin{cases}p_{00}=0, r=0, k=0, m=0;\\ p_{10}=1/2, r=1, k=1, m=0;\\ p_{11}=5/8, r=2, k=1, m=1;\\ p_{20}=6/8, r=2, k=2, m=0;\\ p_{21}=7/8, r=3, k=2, m=1;\\ p_{22}=1, r=4, k=2, m=2.\end{cases}$$

This estimate is not the only one. The second variant of parameter p estimation for the case of $n=2$ [1]:

$$\hat{w}_2\left(r\right):\hat{w}_2\left(0\right)=0;\ \hat{w}_2(1)=\frac{1}{2};$$
$$\hat{w}_2(2)=\frac{2}{3};\ \hat{w}_2(3)=\frac{5}{6};\ \hat{w}_2(4)=1.$$

The unbiassed estimate of the probability of failure for the case of n=3 ($\hat{w}_3(r)$) [1]:

$$\hat{w}_3\left(r\right):\hat{w}_3\left(0\right)=0;\ \hat{w}_3\left(1\right)=\frac{1}{3};\ \hat{w}_3\left(2\right)=\frac{1}{2};\ \hat{w}_3\left(3\right)=\frac{9}{14};$$
$$\hat{w}_3\left(4\right)=\frac{65}{84};\ \hat{w}_3\left(5\right)=\frac{75}{84};\ \hat{w}_3\left(6\right)=1.$$

Estimates $\hat{p}$, $\hat{p}_2$, $\hat{w}_2\left(r\right)$, $\hat{w}_3\left(r\right)$ become useless, when it is required to estimate the unknown parameter p not equal to zero and one.

Let us introduce the concept of centered estimate [1, 7] (not to be confused with the central estimate [4]), namely: let the probability of failure estimate (hereinafter referred to as $\hat{v}$) center the probability function (in our case that is $P_{n\sum}(x,y)$ relative to the limit boundaries of its value range). That means that the ranges $[0;\hat{v}]$ and $[\hat{v};1]$ of the values of such estimates with the probability of 0.5 cover the estimated parameter p. Such estimates we will call centered. Let us note that centered estimates for some test plans are close to efficient estimates [7]. In our case the centered estimate $\hat{v}(\beta=0,5)$ is found using formula $P_{n\sum}\left(x,y\right)=\beta=0,5$, where β does not possess confidence probability any more. Let us also note that the distribution law of statistic $\hat{v}$ is defined by the distribution law of random value R, which allows identifying the confidence boundaries.

Out of the definition of centered estimate follows that it defines the lower (upper) confidence limits (hereinafter referred to as LCL (UCL) of the range of unknown parameter p with confidence probability $\gamma=0.5$ or significance level $\alpha=0.5$. On the other hand, any estimate of the LCL (UCL) of an unknown parameter range p can be interpreted as a point estimate of parameter p with a strong downward (upward) bias. The LCL (hereinafter referred to as $\hat{p}_L$) (UCL (hereinafter referred to as $\hat{p}_U$) of the range of unknown parameter p with confidence probability $\gamma=1-\alpha$ is calculated according to formula (the case of monotonous decrease [1]):

$$P_{n\sum}\left(x,y,\hat{p}_L\right)=\gamma,\ P_{n\sum}\left(x,y,\hat{p}_U\right)=\alpha. \quad (3)$$

Let us note that centered estimates are – in terms of their efficiency – close to the best estimates [7-9], and despite the optimistic definition of the centered estimate $\hat{v}(\beta=0,5)$ this estimate is biased with respect to the estimated parameter $L\left(\hat{v}(n;r;\beta=0,5)\right)>0$. However, this bias can be reduced, thus improving the efficiency [9]. For that purpose, it will suffice to minimize functional $L\left(\tilde{v}(n;r)\right)$ by varying the probability value β=0,5+x in formula $P_{n\sum}=0,5+x$, where x>0 is a positive real number. Thus obtained estimate (hereinafter referred to as $\tilde{v}(\beta=0,5+x)$) is already not centered, but its bias is smaller compared to that of the centered estimate $\hat{v}(\beta=0,5)$, and therefore estimate $\tilde{v}(\beta=0,5+x)$ can be expected to have higher efficiency.

Let us note that function $P_{n\sum}$ monotonously decreases as p grows (proven for cases of n<3) [1], therefore equation

$$P_{n\sum} = \beta = 0,5 + x$$

has a unique solution. Let us once again note that probability β does not imply confidence probability and cannot organize a two-sided confidence interval, as its boundaries "overlap" in opposing directions. Probability β is an indicator parameter that discriminates an estimate out of a set of similar ones in terms of the method of construction β≥0,5.

Additionally, the confidence boundary (β≤0,5) represents a point estimate with a strong bias in relation to the estimated parameter. As the confidence probability β>0 grows, the two-sided confidence interval degenerates first into a point, then stops existing. The one-sided confidence interval stops being such as confidence probability β>0,5 grows, as, with high probability β>0,5, will not cover the estimated parameter. The set of estimates with indicator parameter $\tilde{v}(\beta = 0,5 + x)$ becomes a potential carrier of the efficient estimate.

Let us formulate the selection criterion of the efficient estimate of probability of failure (or PNF), construct – on the basis of the formulated criterion – an improved (but biased) failure probability estimation (and therefore, PNF estimation) for a plan of testing with addition for the case of $n > 3$ and choose the efficient estimate out of those available.

Methods of research of dependability indicator estimates

Efficient estimation is based on the integral approach formulated in [6-11]. The integral approach is based on the formulation of the rule of efficient estimate selection $\hat{\theta}_0(n;k,m)$ specified on the vertical sum of absolute (or relative) biases of estimates $\hat{\theta}_0(n;k,m)$ selected out of a certain set based on the distribution law parameter, where n is the number of products initially submitted to testing.

Criterion of selection of efficient estimation for PNF

The criterion of selection of an efficient estimate of the probability of failure (or PNF) at a set of estimates $\hat{\theta}_0(n;k,m)$ is based on the total square of absolute (or relative) biases of the mathematical expectation of estimates $E\hat{\theta}(n;k,m)$ from probability of failure p for all possible values of p, n.

Let τ be the test time of one product, then the selection of the efficient estimate of the probability of failure (or PNF) will only require the notion of bias-efficient estimate and variation of parameter p within 0≤p≤1. Therefore, for the sake of simplicity, as the criterion for obtaining an efficient estimate $\hat{\theta}_0(n;k,m)$ functional (hereinafter referred to as $L(\hat{\theta}(n;k,m))$) is constructed over limited set 1≤n≤I [7-9]:

$$L\left(\hat{\theta}(n;k,m)\right) = \frac{1}{I}\sum_{n=1}^{I}\int_0^1 \left\{E\hat{\theta}(n;k,m) - p\right\}^2 \partial p. \qquad (4)$$

Estimate $\hat{\theta}_0(n;k,m)$, that minimizes functional $L(\hat{\theta}(n;k,m))$ over the given set of estimates, is called the bias-efficient estimate over the given set of biased estimates. Among the estimates, that afford about the same minimum to functional $L(\hat{\theta}(n;k,m))$, we should choose the estimate that has the minimal mean-square deviation (classical definition of the efficient unbiassed estimate [2]). We will call this estimate more efficient in comparison with the selected ones.

For the purpose of selecting the estimates with minimal deviation, a functional is constructed (hereinafter referred to as $D(\hat{\theta}_0(n;k,m))$ based on the accumulation of mathematical expectations of the squares of relative deviations of estimates $\hat{\theta}_0(n;k,m)$ from parameter p for all possible values p, n [7-9]:

$$D\left(\hat{\theta}(n;k,m)\right) = \frac{1}{I}\sum_{n=1}^{I}\int_0^1 E\left\{\hat{\theta}(n;k,m) - p\right\}^2 \partial p. \qquad (5)$$

We will call estimate that affords zero to functional $L(\hat{\theta}_0(n;k,m))=0$ (unbiassed estimate) and minimum to functional $D(\hat{\theta}_0(n;k,m))$ absolutely bias-efficient.

Let us limit the scope of tests 4≤n≤10, which, for highly dependable and complex products is the cost limit. Then formula (4) will be written as:

$$L\left(\hat{\theta}(n;k,m)\right) = \frac{1}{7}\sum_{n=4}^{10}\int_0^1 \left\{E\hat{\theta}(n;k,m) - p\right\}^2 \partial p.$$

While formula (5) will be written as:

$$D\left(\hat{\theta}(n;k,m)\right) = \frac{1}{7}\sum_{n=4}^{10}\int_0^1 E\left\{\hat{\theta}(n;k,m) - p\right\}^2 \partial p.$$

The performed calculations showed that estimate $\hat{v}(\beta = 0,5 + x)$, that minimizes functionals $L(\hat{\theta}(n;k,m))$ and $D(\hat{\theta}(n;k,m))$, corresponds with β=0,5+x=0,5, i.e. $x = 0$ and subsequently $\hat{v}(\beta = 0,5) = \tilde{v}(\beta = 0,5 + x)$.

Table 1 shows the results of substitution into functionals $L(\hat{\theta}(n;k,m))$ and $D(\hat{\theta}(n;k,m))$, in accordance with formulas (1) and (2), of the following probability of failure estimates $\hat{\theta}$: $\tilde{v}, \hat{p}, \overline{p}$ [1], where

$$\overline{p} = \begin{cases} \tilde{v}\left(0, n, \beta = 0,5\right), r = 0; \\ \dfrac{r}{n+k}, \; r > 0. \end{cases}$$

Functionals $L(\hat{\theta}(n;k,m))$ and $D(\hat{\theta}(n;k,m))$ were calculated with the step of $\partial p = 10^{-3}$. Implicit estimates $\tilde{v}$ and $\overline{p}$ were calculated with the accuracy of 10^{-4}. The scope of tests was limited with the range of 4≤n≤10.

Out of Table 1 follows that under the scope of tests 4≤n≤10 estimate $\hat{p}$ and composite estimates dominate and acquire minimal biases.

Out of Table 1 also follows that estimate $\hat{p}$ and composite estimates $\overline{p}$ are almost equal in terms of deviations of their values from parameter p and insignificantly exceed as such estimate $\tilde{v}$. Therefore estimate $\hat{p}$ can be adopted as the desired bias-efficient estimate among the available ones, when the scope of tests is n>3. However, when it is required to

Table 1. Results of the substitution of available probability of failure estimates into functionals $L(\hat{\theta}(n;k,m))$ and $D(\hat{\theta}(n;k,m))$

Type of functional	$\tilde{v}(\beta=0,5)$ $4 \le n \le 10$	$\hat{p}=\frac{r}{n+k}$ $4 \le n \le 10$	$\overline{p}(\tilde{v}(\beta=0,5))$ $4 \le n \le 10$
$L(\hat{\theta}(n;k,m))$	0,00229	0,000219	0,000805
$D(\hat{\theta}(n;k,m))$	0,0205	0,0186	0,0164

estimate unknown parameter p with a value other than zero and one, estimate $\overline{p}$ should be used.

Let us note that, when calculating, variation of the step of summation ($\partial p = 10^{-3}$) modifies the results of the functional, but does not bring essential changes. The result of comparison does not affect the estimates.

Example 1. Products are part of a redundant unit. It is required to perform a point estimation of the products' PNF subject to the results of binomial tests of such products' dependability. While planning determinative dependability tests the tester calculated sample size (N=n+k=5) assuming a single failure (Q=k=1), thus minimizing the risks caused by the occurrence of such unlikely random failure.

The predicted value of PNF was calculated using a bias-efficient composite estimate [9]:

$$\dot{P} = \begin{cases} 1-\tilde{b}\left(0,N,\beta=0,86\right), R=0; \\ 1-\frac{R}{N},\ R>0, \end{cases}$$

where $\tilde{b}\left(0,N,\beta=0,86\right)$ is the implicit estimate of the binomial test plan [9]. The predicted value of PNF was $\dot{P}(r=1)=1-\frac{r}{N}=\frac{4}{5}=0,8$, which complies with the product's performance specification (PNF is to be not less than 0.8). Given that, during the test time, product failure is unlikely, it was decided to conduct dependability testing using addition in order to save costs. The testing can have two outcomes, i.e. failure-free and one failure (planned). In case of failure-free testing, there is no need for testing with oversampling. The calculations of possible PNF values are given in Tables 2 and 3.

Let us note that in case of binomial testing with curtailed sample N=n=4, Q=0 and when one failure r=1 occurs, the rules require retesting according to the same rules, as $\dot{P}=1-\frac{r}{N}=\frac{3}{4}=0,75<0,8$ [3]. Repeated binomial testing does not allow failures. Performing failure-free binomial tests with the acceptance number of failures of Q=1 will require a sample of size N=5, that is larger than the initial sample used in testing with addition N=4.

That is the advantage of testing with addition that allows making conclusions regarding the compliance with specifications based on the results of a single test with different outcomes, i.e. N=n+k=4, r=0 and N=n+k=5, r=1 (without same-scope (N=n+k=4, r=0) testing as in the case of binomial testing, where one failure is allowed Q=0).

Example 2. Per example 1, the tester, while calculating the size of the sample (N=4), made an allowance for one failure (Q=k=1). The predicted value of PNF was $\dot{P}=1-\frac{r}{N}=\frac{3}{4}=0,75$, which complies with the product's performance specification (PNF is to be not less than 0.75). Given that, during the test time, product failure is unlikely, it was decided to conduct dependability testing using addition in order to save costs. The calculations of possible PNF values are given in Tables 4 and 5.

Criterion of selection of efficient estimate for mean time to failure

Let us assume that the products' time to failure follows the exponential distribution law of probabilities (hereinafter referred to as d.l.) with parameter T_0, where the latter is

Table 2. Results of failure-free testing per example 1

PNF (failure-free tests with addition) r=0, n=4, N=n+k=4+0=4, Q=1			PNF (binomial tests) r=0, N=n=4, Q = 0 $\dot{P}=1-\tilde{b}(0,N,\beta=0,86)$
$1-\tilde{v}$ β=0,5 [1]	$1-\hat{p}=1-\frac{r}{n+k}$	$1-\overline{p}\left(\tilde{v}\right)$ β=0,5	
0,871	1	0,871	0,963

Table 3. Results of tests with one failure per example 1

PNF (failure-free tests with addition) r=1, n=4, N=n+k=5, Q = 1			PNF (binomial tests) r=1, N=n=5, Q = 1 $\dot{P}=1-\frac{r}{N}$
$1-\tilde{v}$ β=0,5	$1-\hat{p}$	$1-\overline{p}\left(\tilde{v}\right)$ β=0,5	
0,687	0,8	0,8	0,8

Table 4. Results of failure-free testing per example 2

PNF (failure-free tests with addition) r=0, k=0, n=3, N=n+k=3+0=3, Q=1				PNF (binomial tests) r=0, N=n=3, Q = 0 $\dot{P}=1-\tilde{b}(0,N,\beta=0,86)$
$1-\tilde{v}$, β=0,5	$1-\hat{p}=1-\frac{r}{n+k}$	$1-\hat{w}_3(0)$	$1-\overline{p}(\tilde{v})$, β=0,5	
0,841	1	1	0,841	0,951

Table 5. Results of tests with one failure per example 2

PNF (failure-free tests with addition) r=1, k=1, n=3, N=n+k=4, Q=1				PNF (binomial tests) r=1, N=n=4, Q = 1 $\dot{P}=1-\frac{r}{N}$
$1-\tilde{v}$, β=0,5	$1-\hat{p}$	$1-\hat{w}_3(1)$	$1-\overline{p}(\tilde{v})$, β=0,5	
0,616	0,75	0,642	0,75	0,75

identical to the mean time to failure (hereinafter referred to as MTF). Then the expected value of PNF of one product within the given time τ will be defined by the equation:

$$P_0(\tau)=e^{\left(-\frac{\tau}{T_0}\right)}.$$

As the quality criterion of the obtained efficient estimate of MTF a functional is constructed (hereinafter referred to as $V(\hat{\theta})$), that is based on the sum of the squares of relative biases of mathematical expectations of estimates $\hat{\theta}(k,m,n,\tau)$ relative to parameter t of the exponential d.l. (MTF) for all possible values of t, n [6]:

$$V\left(\hat{\theta}(k,m,n,\tau)\right)=$$
$$=\frac{1}{3}\sum_{i=3}^{5}\frac{1}{10}\sum_{n=1}^{10}\int_0^\infty\left(\frac{1}{t}\right)^2\{E\hat{\theta}(k,m,n,\tau=10^i)-t\}^2\partial t. \quad (3)$$

Integration is performed using all possible values of parameter (MTF) $t\in[0;\infty]$.

Let us examine the functional (hereinafter referred to as $H(\hat{\theta})$) based on the sum of mathematical expectations of the squares of relative deviations of estimates $\hat{\theta}(k,m,n)$ relative to parameter t of the exponential d.l. (MTF) for all possible values of t, n [6]:

$$H\left(\hat{\theta}(k,m,n,\tau)\right)=$$
$$=\frac{1}{3}\sum_{i=3}^{5}\frac{1}{10}\sum_{n=1}^{10}\int_0^\infty\left(\frac{1}{t}\right)^2E\{\hat{\theta}(k,m,n,\tau=10^i)-t\}^2\partial t. \quad (4)$$

The purpose of functionals $H\left(\hat{\theta}(k,m,n,\tau)\right)$ is to identify the scatter of the values of the available estimates.

Estimate that minimizes the available functionals is efficient among the available estimates of MTF.

Selection of the efficient estimate of MTF

Let us define the estimate of MTF ($\hat{T}_2$) for the plan of tests with addition as:

$$\hat{T}_2=\frac{S(k,m,\tau,s_i,n)}{R},$$

where s_i are the instants of failure, i=1,2,…, R>0, S – is the total operation time. Let us complete estimate $\hat{T}_2$ for the case of R = 0 with value $\hat{T}_2$=$S(k,m,\tau,n)$.

Another case. In order to avoid dividing by zero while estimating the MTF $\hat{T}_2$, let us represent it as follows:

$$\hat{T}_3=\frac{S(k,m,\tau,s_i,n)}{R+1}.$$

Let us consider a simple case and reduce the number of variables for estimates $\hat{T}_3$ and $\hat{T}_2$. For that purpose, let us assume that scatter s_i is symmetrical in relation to τ/2. That can be fulfilled for highly dependable products $\frac{\tau}{T_0}<0{,}1$ [3]. Therefore $S(k,m,\tau,n)$= $(n–k)*\tau+(k+m)*\tau/2$.

Let us define the following estimates of MTF for the plan of tests with addition as:

$$\hat{T}_0=\frac{\tau}{-\ln(1-\tilde{v}(k,m,n,\beta=0,5))},$$

$$\hat{T}_1=\frac{\tau}{-\ln(1-\overline{p}(k,m,n,\beta=0,6))}.$$

Functionals $V(\hat{\theta}(\tau;n;k,m))$ and $H(\hat{\theta}(\tau;n;k,m))$ were calculated with the step of $\partial p=10^{-3}$. Implicit estimates $\tilde{v}$ and $\overline{p}$ were calculated with the accuracy of 10^{-4}.

Table 6. Results of substitution into functionals $V(\hat{\theta}(\tau;n;k,m))$ and $H(\hat{\theta}(\tau;n;k,m))$ of MTF estimates: $\hat{T}_0$, $\hat{T}_1$, $\hat{T}_2$, $\hat{T}_3$

Type of functional	$\hat{T}_0(\tilde{v})$ β=0,5	$\hat{T}_1(\overline{p})$ β=0,6	$\hat{T}_2=\frac{S}{R}$	$\hat{T}_3=\frac{S}{R+1}$
$V(\hat{\theta}(\tau;n;k,m))$	10,89	10,80	2363	1836
$H(\hat{\theta}(\tau;n;k,m))$	27,47	25,54	2373	1845

Table 7. Results of failure-free testing per example 3

PNF (failure-free tests with addition)	PNF (binomial tests)
PNF (failure-free tests with addition) r=0, k=0, n=4, N=n+k=4+0=4, Q=1 $\hat{T}_1 = \dfrac{\tau}{-\ln(1-\overline{p}(k=0,m=0,n,\beta=0,5))}$	PNF (binomial tests) r=0, N=n=4, Q = 0 $\hat{T}_{БИ} = \begin{cases} \dfrac{\tau}{-\ln(1-\frac{r>0}{n})} \\ \dfrac{\tau}{-\ln\left(1-\tilde{b}\left(r=0,n,\beta=0,6\right)\right)} \end{cases}$
$\dfrac{10000}{-\ln(1-\tilde{v}(k=0,m=0,n=4,\beta=0,5))} = 72411$	$\dfrac{10000}{-\ln\left(1-\tilde{b}\left(r=0,n=4,\beta=0,6\right)\right)} = 78304$

Table 8. Results of tests with one failure per example 3

PNF (failure-free tests with addition)	PNF (binomial tests)
PNF (failure-free tests with addition) r=1, k=1, n=4, N=n+k=4+1=5, Q=1 $\hat{T}_1 = \dfrac{\tau}{-\ln(1-\overline{p}(k,m,n,\beta=0,5))}$	PNF (binomial tests) r=1, N=n=5, Q = 1 $\hat{T}_{БИ} = \begin{cases} \dfrac{\tau}{-\ln(1-\frac{r>0}{n})} \\ \dfrac{\tau}{-\ln\left(1-\tilde{b}\left(r=0,n,\beta=0,6\right)\right)} \end{cases}$
$\dfrac{10000}{-\ln(1-\frac{k+m}{n+k})} = 44814$	$\dfrac{10000}{-\ln(1-\frac{1}{5})} = 44814$

Table 6 shows the results of substitution into functionals $V(\hat{\theta}(\tau;n;k,m))$ and $H(\hat{\theta}(\tau;n;k,m))$, in accordance with formulas (3) and (4), of the following MTF estimates $\hat{\theta}$: $\hat{T}_0$, $\hat{T}_3$, $\hat{T}_1$, $\hat{T}_2$.

Out of Table 6 follows that estimate

$$\hat{T}_1 = \frac{\tau}{-\ln(1-\overline{p}k,m,n,\beta=0,6))}$$

is efficient out of the available estimates.

Example 3. Per example 1, products were tested during 10 000 hours. Let us use the classical efficient estimate of MTF $\hat{T} = \dfrac{\tau}{-\ln(1-\hat{b}\left(r,n\right))}, \hat{b}\left(r,n\right) = \dfrac{r}{n}$ for binomial plan [7] and efficient estimate of MTF $\hat{T}_b = \dfrac{\tau}{-\ln(1-\tilde{b}(R=0,N,\gamma=0,6))}$ [9], and construct on their basis the following composite estimate of MTF for binomial testing:

$$\hat{T}_{BT} = \begin{cases} \dfrac{\tau}{-\ln(1-\hat{b})}, R>0; \\ \dfrac{\tau}{-\ln\left(1-\tilde{b}\left(R,n,\beta=0,6\right)\right)}, R=0, \end{cases}$$

where $\tilde{b}(r,n,\beta=0,6)$ is the implicit estimate of the probability of failure of the binomial test plan [9].

$$\hat{T}_{BT}(r=1,n=5) = \frac{\tau}{-\ln(1-\hat{b}(r,n))} = $$

$$= \frac{10000}{-\ln(1-\frac{r}{n})} = \frac{10000}{-\ln(1-\frac{1}{5})} = 44814 \text{ hours},$$

which is in compliance with the performance specification ($T_0 \geq 40000$) for the products. Given that during the test time product failure is unlikely, it was decided to conduct dependability testing using addition in order to save costs.

Conclusions

PNF estimates for the plan of tests with addition were examined. For the case of $n>3$, estimates $\hat{P} = 1-\hat{p} = 1-\dfrac{r}{n+k}$ and $1-\overline{p}(\tilde{v}(\beta=0,5))$ (composite estimate) are more efficient in comparison with estimate $1-\tilde{v}(\beta=0,5)$. The composite estimate of PNF $1-\overline{p}(\tilde{v}(\beta=0,5))$ should be used in failure-free tests.

For the case of $n>3$, testing with the acceptance number of failures greater than zero ($Q>0$) conducted with addition allows reducing the number of tested products through successful testing of the original sample.

The composite estimate of MTF $\hat{T}_1 = \dfrac{\tau}{-\ln(1-\overline{p}(k,m,n,\beta=0,6))}$ is bias-efficient among the proposed MTF estimates.

The obtained composite estimates $\overline{p}$ and $\hat{T}_1$ are of practical significance in the context of failure-free testing with addition.

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The authors’ contribution

Mikhailov V.S. constructed efficient estimates for a plan of tests with addition, such as probability of no-failure $\overline{p}$ и and mean time to failure $\hat{T}_1$. The obtained estimates of $\overline{p}$ and $\hat{T}_1$ are practically applicable in tests that produce no failures and are conducted according to a plan of testing with addition