Dependability, vol. 19 no. 4, 2019 Original article
https://doi.org/10.21683/1729-2646-2019-19-4-12-16

Dispersion of the number of failures in restoration processes

Vitaly I. Vaynshteyn, *Siberian Federal University, Russian Federation, 660041, Krasnoyarsk Krai, Krasnoyarsk, 79 pr. Svobodny*

Vitaly I. Vaynshteyn

Abstract. *Optimal organization of the restoration process is of significant importance in the operation of technical, information and computer systems, since failures occurring during their operation lead to substantial negative consequences. In this paper, a formula for the variance of the number of failures is obtained for the general restoration process, which depends on the restoration functions (average number of failures) of the simple and general restoration processes. Also obtained are the formulas for the variances of the number of failures and restorations during the alternating restoration process, when along with the element's time to failure, for example, the restoration time is taken into account. For an exponential distribution with a simple and general restoration process, formulas are written for the variance of the number of failures, as well as the Chebyshev inequality and the formula for the coefficient of variation of the number of failures for a simple restoration process. The paper presents an algorithm for obtaining dispersion in the form of series for the operation time distribution laws common to the dependability theory. The developed mathematics are intended for the definition and solution of various optimization problems of information and computer security, as well as in the operation of technical and information systems, software and formware information protection facilities affected by random failures, threats of attacks and security threats.*

Keywords: *distribution function, restoration process, restoration function, failure number dispersion, variation coefficient.*

For citation: *Vaynshteyn VI. Dispersion of the number of failures in restoration processes. Dependability 2019; 4: 12-16 p. https://doi.org/10.21683/1729-2646-2019-19-4-12-16.*

Received on 03.09.2019 / Revised on 22.10.2019 / For printing 14.12.2019

Introduction. Problem definition. A sequence of non-negative, mutually independent, random values X_i with distribution functions $F_i(t)$ is called a restoration process [1-3]. In the reliability theory, in the process of restoration after each failure an element is repaired or replaced (with a restoration element) and X_i is the element's times to failure after the (i–1)-th restoration, $F_i(t)$ is their distribution function.

Depending on the structure of the sequence of distribution functions $F_i(t)$ there are various models of the restoration process [1-8].

Thus, if all random values X_i have the same distribution function $F_1(t)$, $F_i(t)=F_1(t)$, we have a simple restoration process. If $F_i(t)=F_1(t)$, $i\geq 2$, we have a general restoration process.

The restoration process defines the random value $N(t)$, i.e. the number of failures (restorations) over the time from 0 to t

$$P\big(N(t)=n\big)=F^{(n)}(t)-F^{(n+1)}(t), \qquad (1)$$

$F^{(n)}(t)$ is the n-fold comparison of distribution functions $F_i(t)$, i=1,2,…,n

$$F^{(n)}(t)=(F^{(n-1)}*F_n)(t)=\int_0^t F^{(n-1)}(t-x)dF_n(x), F^{(1)}(t)=F_1(t).$$

Of significant importance as regards the theoretical and practical problems of the dependability theory is the restoration function $H(t)$, i.e. the mathematical expectation of the number of failures over the time from 0 to t in the process of restoration $H(t)=E(N(t))$

$$H(t)=\sum_{n=1}^{\infty} nF^{(n)}(t). \qquad (2)$$

Let $HF_1(t)$ be the restoration function of a simple process shaped by the distribution function $F_1(t)$, $HF_1F_2(t)$ be the restoration function of the general process shaped by the first distribution function $F_1(t)$, as well as the second and the subsequent $F_2(t)$.

The restoration function $HF_1(t)$ of a simple process satisfies the integral equation

$$HF_1(t)=F_1(t)+\int_0^t HF_1(t-x)dF_1(x). \qquad (3)$$

The restoration function of a general restoration process is expressed through the restoration function of a simple process using formula

$$HF_1F_2(t)=F_1(t)+\int_0^t HF_2(t-x)dF_1(x).$$

For a simple restoration process, the formula for calculating the failure number dispersion is known [2]

$$D(N(t))=2\int_0^t HF_1(t-x)dHF_1(x)+HF_1(t)-H^2F_1(t). \qquad (4)$$

Further study aims to obtain the formula for the failure number dispersion under a general restoration process and development of the method of its calculation for various distribution laws for the operation times of replaced failed elements.

Calculation of dispersion for general restoration processes.

By definition

$$D(N(t))=E(N^2(t))-E^2(N(t))=E(N^2(t))-H^2(t).$$

Thus, in order to calculate the dispersion, along with the restoration function $E(N^2(t))$ must be calculated. Let us examine the calculation of $E(N^2(t))$ [9].

Considering (1), (2) we obtain

$$E(N^2(t))=\sum_{n=1}^{\infty} n^2 P(N(t)=n)=\sum_{n=1}^{\infty} n^2(F^{(n)}(t)-F^{(n+1)}(t))=$$
$$=F_1(t)+\sum_{n=2}^{\infty}(n^2-(n-1)^2)F^{(n)}(t)=F_1(t)+\sum_{n=2}^{\infty}(2n-1)F^{(n)}(t)=$$
$$=-H(t)+2F_1(t)+2\sum_{n=2}^{\infty} nF^{(n)}(t)=-H(t)+2\sum_{n=1}^{\infty} nF^{(n)}(t).$$

Thus, the problem comes down to calculating the sum $\sum_{n=1}^{\infty} nF^{(n)}(t)$ for each model of the restoration process. Further, this sum will be calculated using restoration function formula (2) and definition of the general restoration process. Successively, we obtain

$$\sum_{n=1}^{\infty} nF^{(n)}(t)=F_1(t)+2(F_1*F_2)(t)+3(F_1*F_2^{(2)})(t)+\ldots+$$
$$+n(F_1*F_2^{(n-1)})(t)+\ldots=(F_1(t)+(F_1*F_2)(t)+$$
$$+(F_1*F_2^{(2)})(t)+\ldots+(F_1*F_2^{(n-1)})(t)+\ldots)+((F_1*F_2)(t)+$$
$$+(F_1*F_2^{(2)})(t)+(F_1*F_2^{(3)})(t)+\ldots+(F_1*F_2^{(n-1)})(t)+\ldots)+$$
$$+((F_1*F_2^{(2)})(t)+(F_1*F_2^{(3)})(t)+\ldots+(F_1*F_2^{(n-1)})(t)+\ldots)+$$
$$+(F_1*F_2^{(3)})(t)+\ldots+(F_1*F_2^{(n-1)})(t)+\ldots)+\ldots+$$
$$+(F_1*F_2^{(n-1)})(t)+(F_1*F_2^{(n)})(t)+\ldots)=$$
$$+(F_1*F_2^{(3)})(t)+\ldots+(F_1*F_2^{(n-1)})(t)+\ldots)+\ldots+$$
$$6(F_1*F_2^{(n-1)})(t)+(F_1*F_2^{(n)})(t)+\ldots)=(H(t)+$$
$$+(F_1*HF_2)(t)+(F_1*F_2*HF_2)(t)(F_1*F_2*F_2*HF_2)(t)+$$
$$+\ldots+(F_1*(F_2^{(n)}*HF_2)(t)+\ldots)=H(t)+$$
$$+(HF_2*(F_1+(F_1*F_2)(t)+((F_1*F_2*F_2)(t)+\ldots+$$
$$+(F_1*F_2^{(n)})(t)+\ldots))=H(t)+(HF_2*H)(t).$$

Here $H(t)=HF_1F_2(t)$. Finally,

$$D(N(t))=2\int_0^t HF_2(t-x)dHF_1F_2(x)+HF_1F_2(t)-(HF_1F_2(t))^2. \qquad (5)$$

An example. Let us write the dispersions for the simple and general processes under an exponential distribution of operation times

$$F_1(t)=(1-e^{-\alpha_1 t}),\quad F_2(t)=(1-e^{-\alpha_2 t}),\alpha_1\neq\alpha_2.$$

Under a simple process, $H(t)=\alpha_1 t$. Upon integration in (4), $D(N(t))=\alpha_1 t$

Under a simple process and exponential distribution of operation times the dispersion matches the restoration function.

Under a general process [3].

$$HF_1F_2(t)=\alpha_2 t+(1-\frac{\alpha_2}{\alpha_1})(1-e^{-\alpha_1 t}).$$

Upon integration in (5),

$$D\big(N(t)\big)=\alpha_2^2t^2+\frac{2\alpha_2(\alpha_1-\alpha_2)}{\alpha_1}t-$$
$$-\frac{2\alpha_2(\alpha_1-\alpha_2)}{\alpha_1^2}\big(1-e^{-\alpha_1 t}\big)+HF_1F_2(t)-H^2F_1F_2(t).$$

For many know distribution laws that are common to the dependability theory [10], for instance, exponential, Weibull-Gnedenko, Erlang, normal, Maxwell, Raileigh, gamma and their combinations, the restoration function is obtained in an explicit form or expressed as power series [2, 3, 11, 12].

In [3, 12], it is noted that the above distribution functions and their combinations are expanded into power series as follows

$$F(t)=\sum_{n=0}^{\infty}a_n t^{\beta n+\gamma},\qquad \gamma\geq 0,\ \beta>0. \tag{6}$$

That enables the development of a single algorithm for finding the restoration function of simple processes formed by the distribution function of type (6), provided that numbers β and γ are whole, non-negative or related as $\gamma=l\beta$, l is whole, non-negative. In this case, the restoration function is defined as the solution of the corresponding integral equation (3), if the solution is sought in the following form:

$$H(t)=\sum_{n=0}^{\infty}c_n t^{\beta n+\gamma}. \tag{7}$$

Coefficients c_n are identified.

In [12], in a similar way restoration functions are found for combinations of the above distribution functions, except for the combination of gamma distributions. Under non-natural values of γ, the condition $\gamma=l\beta$, ($\beta=1$) is not fulfilled.

The obtained formulas include integrals $\int_0^t H_1(t-x)dH_2(x)$ of restoration functions for the purpose of dispersion calculation. Let in accordance with (7)

$$H_i(t)=\sum_{n=0}^{\infty}c_{i,n}t^{\beta_i n+\gamma_i},i=1,2.$$

We have

$$\int_0^t H_1(t-x)dH_2(x)=$$
$$=\int_0^t\Big(\sum_{n=0}^{\infty}c_{1,n}(t-x)^{\beta_1 n+\gamma_1}\sum_{k=0}^{\infty}c_{2,k}(\beta_2 k+\gamma_2)x^{\beta_2 k+\gamma_2-1}\Big)dx=$$
$$=\sum_{k=0}^{\infty}c_{2,k}(\beta_{2,k}+\gamma_2)\sum_{n=0}^{\infty}c_{1,n}\int_0^t(t-x)^{\beta_1 n+\gamma_1}x^{\beta_{2,k}+\gamma_2-1}\Big)dx=$$
$$=\sum_{k=0}^{\infty}c_{2,k}\left(\beta_2+\gamma_2\right)\sum_{n=0}^{\infty}c_{1,n}t^{(\beta_1 n+\beta_2 k+\gamma_1+\gamma_2)}\cdot$$
$$\cdot\frac{\Gamma(\beta_1 n+\gamma_1+1)\Gamma(\beta_2 k+\gamma_2)}{\Gamma(\beta_1 n+\beta_2 k+\gamma_1+\gamma_2+1)}. \tag{8}$$

The following was taken into consideration:

$$\int_0^t(t-x)^{\alpha}x^{\beta}dx=t^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)},$$

$\Gamma(x)=\int_0^{\infty}t^{x-1}e^{-t}dt$ is a gamma function.

If $\beta_1=\beta_2=\beta$, then (8) can only arrive to a single infinite sum by replacing $n+k+s$

$$\int_0^t H_1(t-x)dH_2(x)=$$
$$=\sum_{s=0}^{\infty}\frac{t^{\beta s+\gamma_1+\gamma_2}}{\Gamma(\beta s+\gamma_1+\gamma_2+1)}\sum_{n+k=s}c_{1,n}c_{2,k}\Gamma\cdot$$
$$\cdot(\beta n+\gamma_1+1)\Gamma(\beta k+\gamma_2)=$$
$$=\sum_{n=0}^{\infty}\frac{t^{\beta n+\gamma_1+\gamma_2}}{\Gamma(\beta n+\gamma_1+\gamma_2+1)}\sum_{k=0}^{n}c_{2,k}c_{1,n-k}\Gamma\cdot$$
$$\cdot(\beta(n-k)+\gamma_1+1)\Gamma(\beta k+\gamma_2).$$

The definition of the restoration process assumed that the restoration of a failed element happens instantaneously. In practice, this assumption is often false. Along with the time of no-failure, of significant potential importance is the down time, cause of failure identification time and restoration time itself.

Let us examine the so-called simple alternating restoration process [2, 3].

Let (X_n), (Y_n) be two sequences of non-negative, mutually independent, random values each of which forms a simple restoration process with distribution functions $F(t)$, $G(t)$ respectively. Sequence (X_n, Y_n) is called simple alternating restoration process [2, 3].

If Y_n is the time of an element's restoration after the n-th failure, X_n is the element's operation time after the $(n-1)$-th restoration (restoration begins after the first failure), then at moments

$$T_1=X_1,\quad T_2=X_1+Y_1+X_2,\ldots,$$
$$T_n=X_1+Y_1+X_2+\cdots+Y_{n-1}+X_n,\ldots$$

failures occur, while at moments

$$S_1 = X_1 + Y_1, S_2 = X_1 + Y_1 + X_2 + Y_2, \ldots,$$
$$S_n = X_1 + Y_1 + X_2 + Y_2 + \cdots + X_n + Y_n, \ldots$$

restorations end.

The times between failures (accounting for the restoration time) form the general restoration process that is defined by the first $F(t)$ and second $(F*G)(t)$ distribution functions. The times between restorations form a simple restoration process with distribution function $(F*G)(t)$ [2, 3].

The average number of failures and average number of restorations are defined by the restoration functions $H_0(t) = HF(F*G)(t),\ H_1(t) = H(F*G)(t)$ respectively.

Let $D_0(t)$ be the failure number dispersion, and $D_1(t)$ be the restoration number dispersion. In accordance with (4, 5), let us write the formulas for dispersions

$$D_0(t) = 2\int_0^t H(F*G)(t-x)dHF(F*G)(x) + H_0(t) - H_0^2(t),$$

$$D_1(t) = 2\int_0^t H(F*G)(t-x)dH(F*G)(x) + H_1(t) - H_1^2(t).$$

Let us note that the value of the restoration function and dispersion of the number of failures enables the solution of various practical problems involving variation coefficients and Chebyshev inequality.

Let us write the variation coefficient $V(N(t))$ and Chebyshev inequality for the restoration process

$$V\big(N(t)\big) = \frac{\sigma\big(N(t)\big)}{H(t)},$$

($\sigma(N(t))$ is the mean square deviation)

$$P(|N(t) - H(t)| \geq \int) \leq \frac{D(N(t))}{\int^2}.$$

Let us examine the simple restoration process under exponential distribution of operation times $F(t)=1-e^{-\alpha t}$. In this case $H(t)=\alpha t$, $D(N(t))=\alpha t$ and

$$V\big(N(t)\big) = \frac{1}{\sqrt{\alpha t}}.$$

As the time of operation increases, the variation coefficient decreases.

Taking $\int = 3\sqrt{D(N(t))}$ and transitioning to the contrary event in the Chebyshev inequality, we obtain a well-known form of the Chebyshev inequality, that in the case of the restoration process becomes

$$P\Big(\big|N(t) - H(t)\big| < 3\sqrt{D\big(N(t)\big)}\Big) \geq \frac{8}{9}.$$

For a simple restoration process under exponential distribution of operation times

$$P(|N(t) - \alpha t| < 3\sqrt{\alpha t}) \geq \frac{8}{9}.$$

Conclusion. The operation of technical and information systems, as well as information security software and firmware is associated with failures, threats of attack, safety threats and many other effects, random in their nature, that negatively affect their operation. Such effects cause restoration processes. The number of failures, threats of attack and safety threats are random values that depend on the time and their distribution functions. The variation patterns of such distribution functions cause the variety in the models of restoration process, for which methods have been developed for finding the mathematical expectation (restoration function) of the failure number.

For the general and alternating restoration processes the paper obtained the formula of dispersion that depends on the restoration function of two processes, the simple and the general. It also suggests an algorithm for calculating the restoration function for operation time distribution functions common to the dependability theory. As an example, dispersion expressions were obtained for the simple, general process under an exponential distribution. For that case, a Chebyshev inequality and variation coefficient were written.

Let us note that obtaining the failure number dispersion formulas for other models of restoration process is of interest as well.

The availability of formulas for the average and dispersion of the failure number, as well as accounting for the joint variation of the average and dispersion of the failure number in the process of restoration from the times to failure distribution functions of restorable elements naturally entails the consideration of new optimization problems in restoration processes. For instance, minimization of the failure dispersion under a restricted value of the average failure number in operation leads to a problem that in terms of its definition is similar to the Markowitz problem of optimal investment portfolio [13, 14].

Thus, the mathematics developed in this paper will find their application in the definition and solution of various optimization problems of information and computer security, as well as in the operation of technical, information, socio-economic, biological and other systems when the occurrence of failures is random.

References

[1] Borovkov AA. Teoriya veroyatnostey [Probability theory]. Moscow: Librokom; 2009 [in Russian].https://elibrary.ru/author_items.asp?refid=376420326&fam=%D0%91%D0%BE%D1%80%D0%BE%D0%B2%D0%BA%D0%BE%D0%B2&init=%D0%90+%D0%90

[2] Beichelt F, Franken P. Reliability and Maintenance. Mathematical Methods. Moscow: Radio i sviaz; 1988.

[3] Vaynshteyn II. Protsessy i strategii vosstanovleniya s izmenyayushchimisya funktsiyami raspredeleniya v teorii

nadezhnosti [Processes and strategies of restoration with ramp distribution functions in the dependability theory]. Krasnoyarsk: SFU; 2016 [in Russian].

[4] Vaynshteyn II, Vaynshteyn VI, Veysov EA. O modelyakh protsessov vosstanovleniya v teorii nadezhnosti [On the models of restoration processes in the dependability theory]. In: Polovinkin VI, editor. Voprosy matematicheskogo analiza: sb. nauch. tr. Vyp. 6 [Matters of mathematical analysis: a collection of research papers. Vol. 6]. Krasnoyarsk: IPTs KGTU; 2003 [in Russian].https://elibrary.ru/author_items.asp?refid=535409857&fam=%D0%92%D0%B0%D0%B9%D0%BD%D1%88%D1%82%D0%B5%D0%B9%D0%BD&init=%D0%98+%D0%98https://elibrary.ru/author_items.asp?refid=535409857&fam=%D0%92%D0%B0%D0%B9%D0%BD%D1%88%D1%82%D0%B5%D0%B9%D0%BD&init=%D0%92+%D0%98 https://elibrary.ru/author_items.asp?refid=535409857&fam=%D0%92%D0%B5%D0%B9%D1%81%D0%BE%D0%B2&init=%D0%95+%D0%90

[5] Bulinskaya EV, Sokolova AI. Asimptoticheskoe povedenie nekotorykh stokhasticheskikh sistem khraneniya [Asymptotic behaviour of some stochastic storage systems]. Sovremennye problemy matematiki i mekhaniki; 2015. p. 37-62 [in Russian]. https://elibrary.ru/author_items.asp?refid=376420328&fam=%D0%91%D1%83%D0%BB%D0%B8%D0%BD%D1%81%D0%BA%D0%B0%D1%8F&init=%D0%95+%D0%92 https://elibrary.ru/author_items.asp?refid=376420328&fam=%D0%A1%D0%BE%D0%BA%D0%BE%D0%BB%D0%BE%D0%B2%D0%B0&init=%D0%90+%D0%98

[6] Ankudinov AV, Antonov AV, Chepurko VA. Renewal equation for Kijima-Sumita processes. Dependability 2018;18(2):3-9.

[7] Chumakov IA, Antonov AV, Chepurko VA. On some properties of Kijima incomplete restoration models. Dependability 2015;3(54):10-15.

[8] Pereguda AI, Pereguda AA, Timashev DA. The mathematical model of computer networks' reliability. Dependability 2013;(4):31-43.

[9] Vaynshteyn II, Vaynshteyn VI. Dispersiya chisla otkazov v modelyakh protsessov vosstanovleniya tekhnicheskikh i informatsionnykh sistem. Optimizatsionnye zadachi [Dispersion of failure numbers in restoration process models of technical and information technology systems]. Modeling, optimization and information technology 2019; 7(3) [in Russian].

[10] Litvinenko RS, Pavlov PP, Idiyatullin RG. Practical application of continuous distribution laws in the theory of reliability of technical systems. Dependability 2016;16(4):17-23.

[11] Vaynshteyn VI, Veysov EA, Shmidt OO. Chislennoe nakhozhdenie funktsii vosstanovleniya dlya odnoy modeli protsessa vosstanovleniya [Numerical method of finding the restoration function for one restoration process model]. Novosibirsk: Vychislitelnye tekhnologii 2005;10:4-9 [in Russian].https://elibrary.ru/author_items.asp?refid=535409856&fam=%D0%92%D0%B0%D0%B9%D0%BD%D1%88%D1%82%D0%B5%D0%B9%D0%BD&init=%D0%92+%D0%98https://elibrary.ru/author_items.asp?refid=535409856&fam=%D0%92%D0%B5%D0%B9%D1%81%D0%BE%D0%B2&init=%D0%95+%D0%90https://elibrary.ru/author_items.asp?refid=535409856&fam=%D0%A8%D0%BC%D0%B8%D0%B4%D1%82&init=%D0%9E+%D0%9E https://elibrary.ru/contents.asp?titleid=8610

[12] Vaynshteyn VI. Recovery functions of elements of technical systems which operation time distribution is a mixture of n functions distributions. Modern High Technologies 2018;6:44-49 [in Russian].https://elibrary.ru/item.asp?id=35197333https://elibrary.ru/author_items.asp?refid=535409856&fam=%D0%92%D0%B0%D0%B9%D0%BD%D1%88%D1%82%D0%B5%D0%B9%D0%BD&init=%D0%92+%D0%98

[13] Markowitz HM. Portfolio Selection. Journal of Finance 1952;7(1):71-91.

[14] Kasimov YuF. Osnovy teorii optimalnogo portfelya tsennykh bumag [Fundamentals of the theory of an optimal investment portfolio]. Moscow: Filin; 1998 [in Russian].

About the author

Vitaly I. Vaynshteyn, Candidate of Physics and Mathematics, Siberian Federal University, Associate Professor, Head of Laboratory for Information Security, Department of Applied Mathematics and Computer Security, Russian Federation, Krasnoyarsk Krai, Krasnoyarsk, e-mail: vit037@mail.ru

The author's contribution

The paper sets forth a formula for the dispersion of the number of failures of the general restoration process and formula for the dispersion of the number of failures and number of restorations under an alternating restoration process. An algorithm for obtaining the dispersion of the number of failures in the form of series for the laws of operation time distribution common to the dependability theory.